The Bohr & de-Broglie model of the quantum atom

A simple 'planetary' model of an atom, the simplest being Hydrogen with a single proton and a single electron, *cannot* be correct. If the negatively charged electron is executing circular motion about the positively charged nucleus it is *accelerating*. Accelerating charges result in *radiation* of electromagnetic waves, which will *remove energy from the system*. One can show that this means atoms like Hydrogen *should only exist for a tiny fraction of a second.* This is not what we observe in practice! Also, *spectroscopic studies* show that atoms can only absorb and re-radiate energy at *specific frequencies.* This means the **energies** of the electron (and or the nucleus) can also only be at **discrete values**. At the turn of the twentieth century a new model was required. The model below forms the basis of the 'old quantum theory' (proposed by Bohr). We will use de Broglie's ideas of *wave-particle duality* in this model from the outset, which Bohr didn't use at the time. (He started from *quantized angular momentum*).

$$\varepsilon_0 = 8.854187817\times10^{-12}\,\mathrm{Fm^{-1}}$$
$$e = 1.6021766208(98)\times10^{-19}\,\mathrm{C}$$
$$c = 2.99792458\times10^{8}\,\mathrm{ms^{-1}}$$
$$h = 6.626070040(81)\times10^{-34}\,\mathrm{kgm^2s^{-1}}$$
$$m_e = 9.10938356(11)\times10^{-31}\,\mathrm{kg}$$

Niels Bohr
1885-1962
Nobel Prize 1922

Louis de Broglie
1892 – 1987
Nobel Prize 1929

The Bohr model of an atom – a *hybrid* of Quantum and *Classical physics*

The coulomb force F acting upon an electron 'orbiting' at radius r from a nucleus of charge Ze is

$$F = \frac{Ze^2}{4\pi\varepsilon_0 r^2}$$

If an electron executes circular motion about the nucleus , Newton's Second law allows us to determine the magnitude of the acceleration of the electron:

$$m_e a = F$$
$$\therefore a = \frac{v^2}{r} = \frac{Ze^2}{4\pi\varepsilon_0 m_e r^2}$$

The kinetic energy of the electron is therefore:

$$\tfrac{1}{2}m_e v^2 = \frac{Ze^2}{8\pi\varepsilon_0 r}$$

Therefore the total energy is:

$$E = \tfrac{1}{2}m_e v^2 - \frac{Ze^2}{4\pi\varepsilon_0 r}$$

← potential energy due to electrostatic attraction to nucleus

$$E = -\frac{Ze^2}{8\pi\varepsilon_0 r}$$

The radiative power of an accelerating electron is given by:

$$\dot{E} = \frac{dE}{dt} = -\frac{e^2}{6\pi\varepsilon_0 c^3}a^2$$

Using the expression for acceleration:

$$\dot{E} = -\frac{e^2}{6\pi\varepsilon_0 c^3}\times\left(\frac{Ze^2}{4\pi\varepsilon_0 m_e r^2}\right)^2$$
$$\therefore \dot{E} = -\frac{Z^2e^6}{96\pi^3\varepsilon_0^3 c^3 m_e^2 r^4}$$

We can therefore calculate a *characteristic timescale* corresponding to the 'lifetime' of a classically 'orbiting' electron*

$$\tau = \frac{\tfrac{1}{2}m_e v^2}{|\dot{E}|}$$
$$\tau = \frac{Ze^2}{8\pi\varepsilon_0 r}\times\frac{96\pi^3\varepsilon_0^3 c^3 m_e^2 r^4}{Z^2e^6}$$
$$\tau = \frac{12\pi^2\varepsilon_0^2 c^3 m_e^2 r^3}{Ze^4}$$
$$\tau \approx 4.7\times10^{-11}\,\mathrm{s}$$

So according to classical Physics, no atoms should exist!

Louis de Broglie proposed a general '**wave-particle duality.**' All particles with momentum p have an 'associated wave' of wavelength

$$\lambda = \frac{h}{p}$$
$$p = m_e v$$
$$\therefore m_e v = \frac{h}{\lambda}$$

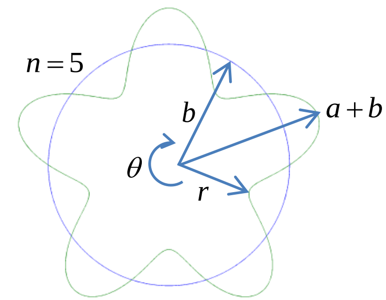
In a stable atom, it would make sense for electrons to be represented as *standing waves*. Therefore:

$$2\pi r = n\lambda$$
$$\lambda = \frac{2\pi r}{n}$$

where n is a positive integer. Using the de Broglie relationship:

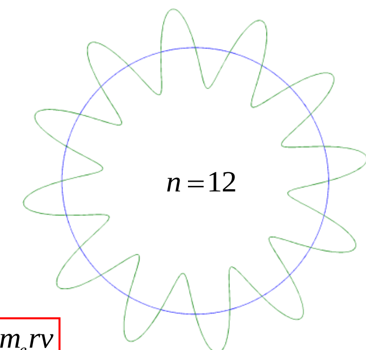
$$m_e v = \frac{h}{\lambda}$$
$$\therefore m_e v = \frac{nh}{2\pi r}$$
$$\therefore m_e r v = n\hbar \qquad \hbar = \frac{h}{2\pi}$$

This last expression is the **angular momentum** of the electron, which appears to be *quantized*. $L = m_e r v$



'Circular sine waves' of the form

$$r = a\sin n\theta + b$$
$$n\lambda = 2\pi b \quad \text{for waves to 'fit'}$$
$$\therefore r = a\sin\left(2\pi\times\frac{b\theta}{\lambda}\right) + b$$



*An improved calculation would factor in the inward spiralling of the orbit as the electron loses energy

From the quantizing of electron waves and the **de Broglie relationship**, the electron orbital velocity is:

$$v = \frac{n\hbar}{m_e r}$$

Using **Newton's Second Law** for the orbiting electron

$$\frac{m_e v^2}{r} = \frac{Ze^2}{4\pi\varepsilon_0 r^2}$$

$$\therefore \frac{m_e}{r}\left(\frac{n\hbar}{m_e r}\right)^2 = \frac{Ze^2}{4\pi\varepsilon_0 r^2}$$

$$\frac{n^2\hbar^2}{m_e r} = \frac{Ze^2}{4\pi\varepsilon_0}$$

Electron **orbital radii** are therefore **quantized**

$$\therefore r_n = \frac{4\pi\varepsilon_0 n^2\hbar^2}{m_e Ze^2} = n^2 r_1$$

$$r_1 = \frac{4\pi\varepsilon_0\hbar^2}{m_e Ze^2} \approx 0.0529\text{nm}$$

This is called the Bohr radius, if $Z = 1$

Electron **orbital velocity** is *also* quantized

$$v_n = \frac{n\hbar}{m_e r_n}$$

$$v_n = \frac{1}{n}\frac{\hbar}{m_e r_1} = \frac{1}{n}\frac{\hbar m_e Ze^2}{m_e 4\pi\varepsilon_0\hbar^2} = \frac{Ze^2}{4\pi\varepsilon_0\hbar}\frac{1}{n}$$

$$v_n = \frac{\alpha Z}{n}c$$

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} \approx \frac{1}{137}$$

This is called the **Fine Structure Constant**
Note it is dimensionless!

This means electron velocities are *approaching* speeds where **relativistic corrections** need to be applied. We would therefore expect some deviations from this hybrid-classical model. However, unless Z is large, we might anticipate these deviations to be small.

Energy of Bohr orbits, and Spectroscopy

$$E_n = -\frac{Ze^2}{8\pi\varepsilon_0 r_n}$$

From the classical orbit model

$$\therefore E_n = -\frac{Ze^2}{8\pi\varepsilon_0} \times \frac{m_e Ze^2}{4\pi\varepsilon_0 n^2\hbar^2} \leftarrow r_n = \frac{4\pi\varepsilon_0 n^2\hbar^2}{m_e Ze^2}$$

$$E_n = -\frac{m_e Z^2 e^4}{8\varepsilon_0^2 h^2}\frac{1}{n^2} \approx \frac{-13.6\text{eV}}{n^2}$$

When an atom absorbs and the re-radiates light ('photons'), this must correspond to a *change in energy level.* Since this must take discrete values, this means the **emission spectrum of a element must be composed of specific wavelengths** and not a continuous sweep of values.

$E = hf$
photon energy

$$\therefore hf_{nm} = E_n - E_m$$

i.e. energy level change from n to m

$$hf_{nm} = \frac{m_e Z^2 e^4}{8\varepsilon_0^2 h^3}\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$

$$f_{nm} = \frac{m_e Z^2 e^4}{8\varepsilon_0^2 h^3}\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$

$$c = f_{nm}\lambda_{nm}$$

$$\lambda_{nm} = \frac{c}{f_{nm}}$$

$$\lambda_{nm} = \frac{8\varepsilon_0^2 h^3 c}{m_e Z^2 e^4}\left(\frac{1}{m^2} - \frac{1}{n^2}\right)^{-1}$$

$$\lambda_{\infty 1} = \frac{8\varepsilon_0^2 h^3 c}{m_e Z^2 e^4} \approx \frac{91.13\text{nm}}{Z^2}$$

This is the **Balmer Formula** named after the Swiss Mathematics teacher who proposed its form based upon a numerical analysis of known emission spectra of Hydrogen. The Balmer lines are those in the visible spectrum. Lyman lines are ultra-violet whereas the larger wavelength emissions are in the infra-red range.

Wavelengths of visible colours

λ = 400nm	Violet
λ = 445nm	Indigo
λ = 475nm	Blue
λ = 510nm	Green
λ = 570nm	Yellow
λ = 590nm	Orange
λ = 650nm	Red

Spectral lines of Hydrogen $(Z=1)$

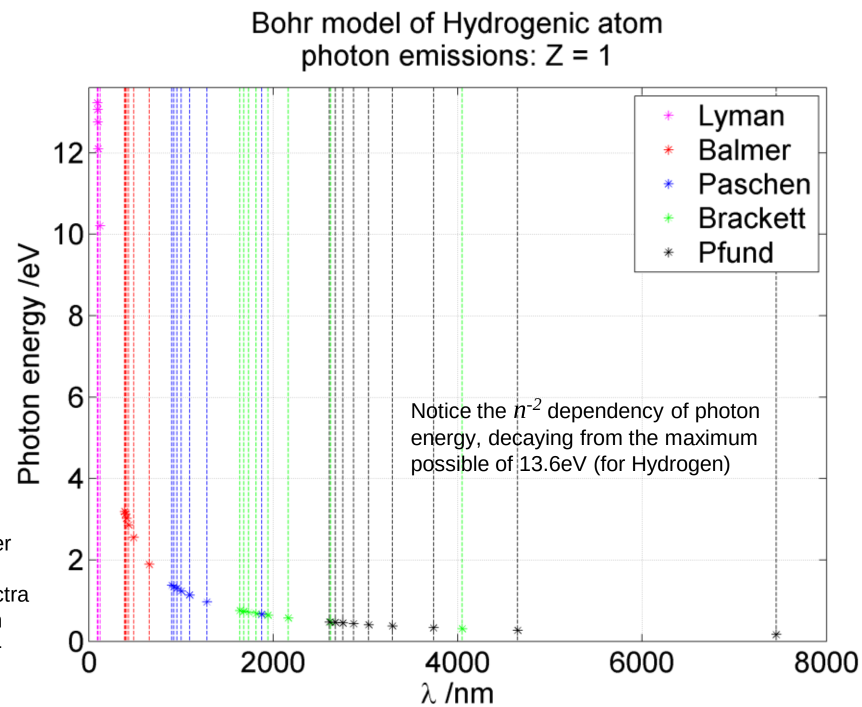
Lyman $n \geq 2, m = 1$

Balmer $n \geq 3, m = 2$

Paschen $n \geq 4, m = 3$

Brackett $n \geq 5, m = 4$

Pfund $n \geq 6, m = 5$



$$1\text{eV} = 1.6021766208(98) \times 10^{-19}\text{J}$$

i.e. one electron-volt. A sensible unit for electron energies. (MeV for *nuclear* Physics).