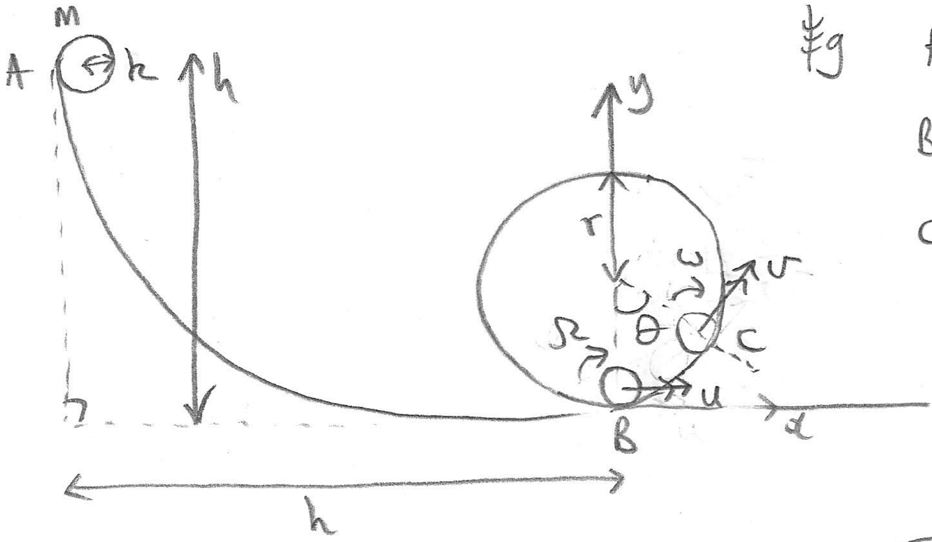
LOOP THE LOOP WITH ROLLING



ENERGY

$\downarrow g$ A: mgh

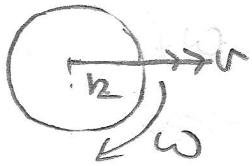
B: $\frac{1}{2}mu^2 + \frac{1}{2}I\Omega^2$

C: $\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mg(r^* - r^*\cos\theta)$

$r - k = r^*$

Assume no slip

So rolling condition is:



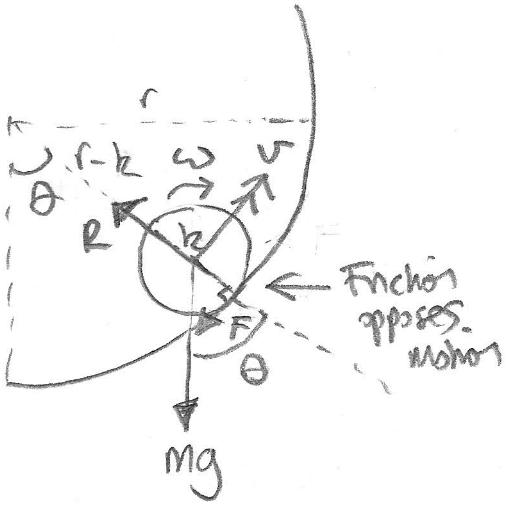
$$\boxed{\begin{aligned} v &= k\omega \\ u &= k\Omega \end{aligned}}$$

[cylinder: $I = \frac{1}{2}mk^2$
sphere: $I = \frac{2}{5}mk^2$]

so $mgh = \frac{1}{2}mk^2\Omega^2 + \frac{1}{2}I\Omega^2$

$\therefore \quad 2mgh = \Omega^2(mk^2 + I)$

$$\therefore \quad \boxed{\sqrt{\frac{2mgh}{mk^2 + I}} = \Omega}$$



NEWTON II

Radially inward to circle

$$\boxed{\frac{mv^2}{r-k} = R - mg\cos\theta}$$

Tangential to circle

$$\boxed{m(r-k)\ddot{\theta} = F + mg\sin\theta}$$

No slip $\boxed{F \leq \mu R}$

$\boxed{\text{Torque} = I\dot{\omega}}$ (about centre of mass of rolling object)

$Fk = I\dot{\omega}$

$\therefore \frac{Fk}{I} = \dot{\omega}$

Now $v = (r-k)\dot{\theta} = k\omega$ ← rolling condition + circular motion

$\therefore (r-k)\ddot{\theta} = k\dot{\omega}$

So $\ddot{\theta}(r-k) = \dfrac{F + mg\sin\theta}{m} = k\dot{\omega} = \dfrac{Fk^2}{I}$

So $\dfrac{F + mg\sin\theta}{m} = \dfrac{Fk^2}{I}$

$$g\sin\theta = F\left(\frac{k^2}{I} - \frac{1}{m}\right)$$

$$mg\sin\theta = F\left(mk^2/I - 1\right)$$

$$\therefore \quad \boxed{F = \frac{mg\sin\theta}{mk^2/I - 1}}$$

Hence since $\dot{\omega} = \dfrac{Fk}{I}$

$$\therefore \quad \boxed{\dot{\omega} = \frac{mgk\sin\theta}{mk^2 - I}}$$

(Non linear equation for θ since $\dot{\omega} = \dfrac{r-k}{k}\ddot{\theta}$)

So for no slip $F \le \mu R$

$R = \dfrac{mv^2}{r-k} + mg\cos\theta$ $\qquad v = k\omega$

So $\dfrac{mg\sin\theta}{mk^2/I - 1} \le \mu\left[\dfrac{mk^2\omega^2}{r-k} + mg\cos\theta\right]$

or $\mu \geq \left(\dfrac{mg\sin\theta}{\frac{mk^2}{I} - 1}\right)\left(\dfrac{mk^2\omega^2}{r-k} + mg\cos\theta\right)^{-1}$

Using energy equation:

$$\frac{1}{2}m\underbrace{k^2\omega^2}_{v^2} + \frac{1}{2}I\omega^2 + mg(r-k)(1-\cos\theta) = mgh$$

$$\omega^2\left(\frac{1}{2}mk^2 + \frac{1}{2}I\right) = mgh - mg(r-k)(1-\cos\theta)$$

$$\omega^2 = \frac{mgh - mg(r-k)(1-\cos\theta)}{\frac{1}{2}mk^2 + \frac{1}{2}I}$$

And when $\theta = 0$, $\omega = \Omega = \sqrt{\frac{2mgh}{mk^2+I}}$

$R \geq 0$ for contact: $R = \frac{mk^2\omega^2}{r-k} + mg\cos\theta$

so $$\frac{mk^2}{r-k}\left[\frac{mgh - mg(r-k)(1-\cos\theta)}{\frac{1}{2}mk^2 + \frac{1}{2}I}\right] + mg\cos\theta \geq 0$$

$$\cos\theta\left(mg + \frac{mg(r-k)}{\frac{1}{2}mk^2+\frac{1}{2}I} \times \frac{mk^2}{r-k}\right) \geq \frac{mg(r-k) - mgh}{\frac{1}{2}mk^2 + \frac{1}{2}I}\frac{mk^2}{r-k}$$

$$\cos\theta \geq \frac{\frac{mg(r-k) - mgh}{\frac{1}{2}mk^2 + \frac{1}{2}I}\frac{mk^2}{r-k}}{mg + \frac{mg\cancel{(r-k)}}{\frac{1}{2}mk^2+\frac{1}{2}I}\frac{mk^2}{\cancel{r-k}}}$$

$$\cos\theta \geq \frac{mg(r-k-h)}{mg\left(\frac{1}{2}mk^2+\frac{1}{2}I\right)\frac{(r-k)}{mk^2} + mg(r-k)}$$

$$\cos\theta \geq \frac{mg(r-k-h)}{mg\left(\frac{1}{2} + \frac{I}{2mk^2}\right)(r-k) + mg(r-k)}$$

Check: $I = 0,\ k = 0$

$$\Rightarrow \quad \cos\theta \geq \frac{mg(r-h)}{mg(\frac{1}{2})r + mgr} = \frac{r-h}{\frac{3}{2}r} = \frac{2}{3}\left(1-\frac{h}{r}\right)$$

For no friction classic problem: $\cos\theta \geq \frac{2}{3}\left(1-\frac{h}{r}\right)$ ✓

$$\text{so} \quad \cos\theta \geq \frac{mg(r-k-h)}{mg(r-k)\left[\frac{1}{2} + \frac{I}{2mk^2} + 1\right]}$$

$$\cos\theta \geq \frac{1 - \frac{h}{r-k}}{\frac{3}{2} + \frac{I}{2mk^2}}$$

$$\boxed{\cos\theta \geq \frac{2}{3}\left(\frac{1-\frac{h}{r-k}}{1+\frac{I}{3mk^2}}\right)}$$

Now $\cos\theta \geq -1$ so $R > 0\ \forall\theta$ if

$$\frac{2}{3}\left(\frac{1-\frac{h}{r-k}}{1+\frac{I}{3mk^2}}\right) < -1$$

$$1 - \frac{h}{r-k} < -\frac{3}{2}\left(1+\frac{I}{3mk^2}\right)$$

$$1 + \frac{3}{2}\left(1+\frac{I}{3mk^2}\right) < \frac{h}{r-k}$$

$$\boxed{\left(\frac{5}{2} + \frac{I}{2mk^2}\right)(r-k) < h}$$

for complete loop the loop.

$\boxed{\text{Cylinder}}$ $I = \frac{1}{2}mk^2$

$\Rightarrow h > \left(\frac{5}{2} + \frac{1}{4}\right)(r-k)$

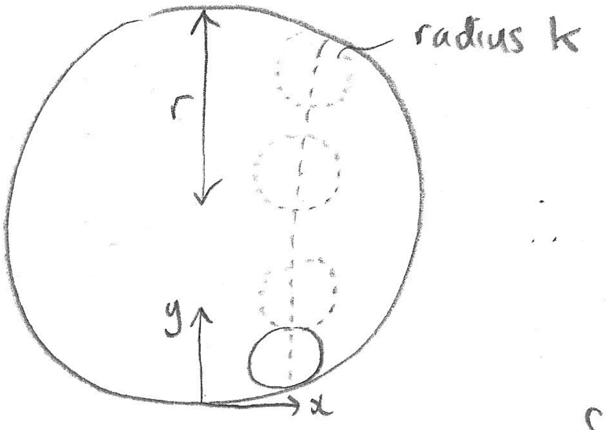
$\Rightarrow \boxed{h > \frac{11}{4}(r-k)}$

$\boxed{\text{Sphere}}$ $I = \frac{2}{5}mk^2$

$\Rightarrow \boxed{h > \frac{27}{10}(r-k)}$

Rolling object becomes a $\boxed{\text{projectile}}$ when $\cos\theta < \frac{2}{3}\left(\dfrac{1-\frac{h}{r-k}}{1+\frac{I}{3mh^2}}\right)$

ie $R < 0$



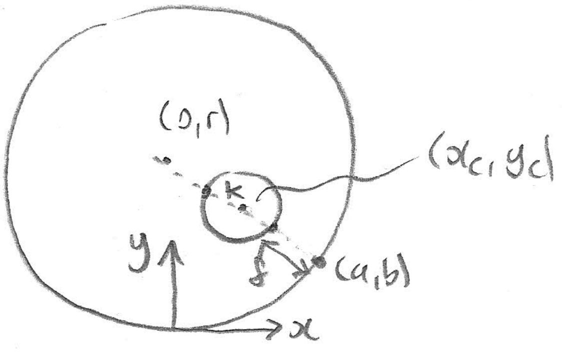
let coordinates of rolling object centre be (x_c, y_c)

$\therefore$ Circle is within (or outside) loop the loop circle of radius r if no solutions for:

$$\begin{cases} (x-x_c)^2 + (y-y_c)^2 = k^2 \\ x^2 + (y-r)^2 = r^2 \end{cases}$$

ie discriminant condition.

Alternatively $\delta > 0$ before $\bigcirc$ collides with loop the loop circle again



$(0,r)$ - - - (x_c, y_c) is

$$y = mx + c$$
$$y_c = mx_c + c$$
$$r = c$$

To go through $(0,r)$ and (x_c, y_c)

So $m = \dfrac{y_c - r}{x_c}$

$$\therefore \boxed{y = \frac{y_c - r}{x_c}x + r}$$

intersects with larger circle when $x^2 + \left(\dfrac{y_c - r}{x_c}x\right)^2 = r^2$

$$x^2\left(1 + \left(\frac{y_c-r}{x_c}\right)^2\right) = r^2$$

$$\therefore \boxed{a = \pm\frac{r}{\sqrt{1+\left(\frac{y_c-r}{x_c}\right)^2}}} \qquad \boxed{b = \frac{y_c-r}{x_c}a + r}$$

Now $(k+\delta)^2 = (x_c-a)^2 + (y_c-b)^2$ so $\boxed{\delta = \sqrt{(x_c-a)^2+(y_c-b)^2} - k}$