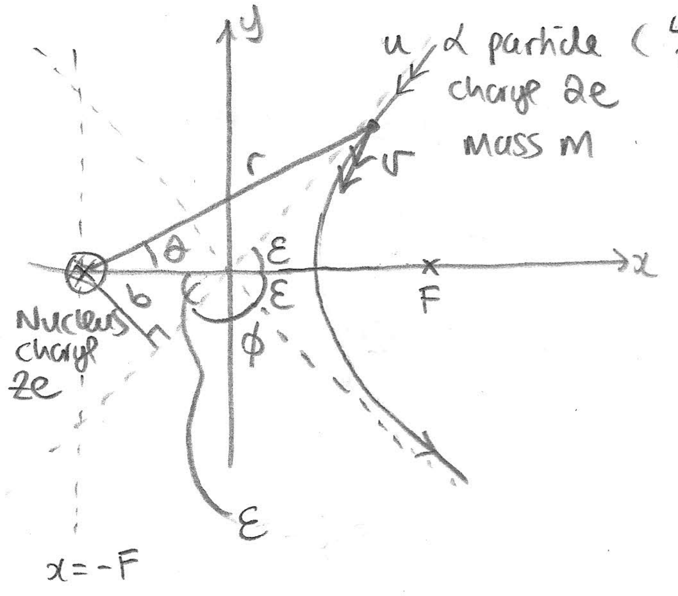
Rutherford Scattering



An α particle, with initial velocity

$$v_\infty = \begin{pmatrix} -u\cos\varepsilon \\ -u\sin\varepsilon \end{pmatrix}$$

is deflected by angle ϕ by a nucleus of charge Ze

e is the |charge| on the electron (or proton!)

$$\boxed{e \simeq 1.6\times10^{-19}\,C}$$

In the absence of any repulsion, the α particle would pass the nucleus at closest approach b

From geometry $\boxed{F\sin\varepsilon = b}$ ①

By conservation of energy *

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + \frac{Ze^2}{4\pi\varepsilon_0 r} \quad ②$$

↑ kinetic energy

↑ Potential energy due to Coulomb repulsion by the charged nucleus

Also note

$$\boxed{\begin{aligned} x &= r\cos\theta - F \\ y &= r\sin\theta \end{aligned}}$$

Now in polar coordinates (r,θ) centred on the nucleus

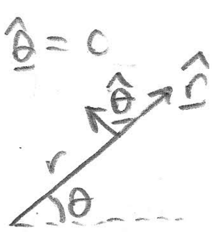
* (assumed to be much more massive, so it is not moved by the proximity of the α particle)

$$\boxed{v^2 = \dot{r}^2 + r^2\dot{\theta}^2}$$

[$\underline{v} = \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}}$

$v^2 = \underline{v}\cdot\underline{v}$, $\hat{\underline{r}}\cdot\hat{\underline{\theta}} = 0$

$|\hat{\underline{r}}| = 1$, $|\hat{\underline{\theta}}| = 1$



$$\therefore\ u^2 = \dot{r}^2 + r^2\dot{\theta}^2 + \frac{Ze^2}{2\pi\varepsilon_0 m r} \quad ③$$

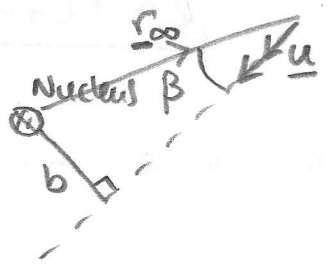
Now since the force on the particle is always in the $\hat{r}$ direction, there can be no net torque. ∴ angular momentum is constant. $\boxed{\underline{L} = m\underline{r} \times \underline{v}}$

In polar coordinates $\underline{r} \times \underline{v} = r\hat{\underline{r}} \times (\dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}})$

$$= r^2\dot{\theta}\,\hat{\underline{r}} \times \hat{\underline{\theta}}$$

$$\therefore \quad L = |\underline{L}| = \boxed{mr^2\dot{\theta}} \qquad (4)$$

So $r^2\dot{\theta}$ is a constant.

Now 

is the initial condition, in relation to the closest approach distance b (which is the 'no deflection') scenario

From the definition of the cross product $|\underline{r}_\infty \times \underline{u}| = r_\infty u \sin\beta = ub$ $\therefore \quad mr^2\dot{\theta} = mub$

$$\boxed{r^2\dot{\theta} = ub} \qquad (5)$$

$\therefore$ in (3) $\quad u^2 = \dot{r}^2 + r^2\left(\frac{ub}{r^2}\right)^2 + \frac{Ze^2}{2\pi\varepsilon_0 Mr} \qquad (6)$

Now let $\boxed{z = \frac{1}{r}}$

$$\frac{dz}{dt} = -\frac{1}{r^2}\frac{dr}{dt} \qquad \therefore \dot{r} = -r^2\frac{dz}{dt}$$

Now $\frac{dz}{dt} = \frac{dz}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{dz}{d\theta} \qquad \therefore \dot{r} = -r^2\dot{\theta}\frac{dz}{d\theta}$

Since $r^2\dot{\theta} = ub \quad \Rightarrow \quad \boxed{\dot{r} = -ub\frac{dz}{d\theta}}$

Hence (6) becomes: $\quad u^2 = u^2b^2\left(\frac{dz}{d\theta}\right)^2 + u^2b^2z^2 + \frac{Ze^2}{2\pi\varepsilon_0 M}z$

Since u is a constant, $\frac{d}{d\theta}(u^2) = 0$

$$\therefore \quad u^2 b^2 \left(2\frac{dz}{d\theta}\right)\frac{d^2z}{d\theta^2} + 2u^2b^2 z\frac{dz}{d\theta} + \frac{Ze^2}{2\pi\varepsilon_0 M}\frac{dz}{d\theta} = 0$$

Since $\frac{dz}{d\theta} \neq 0 \quad \forall\, \theta$!

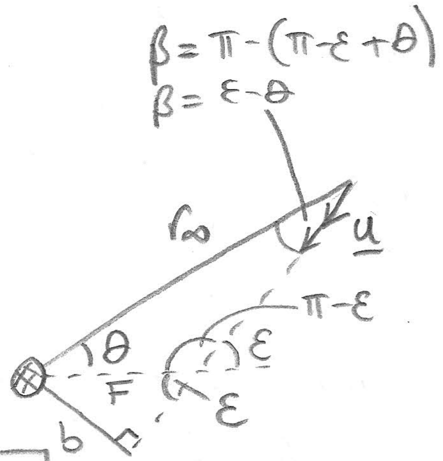
$$\Rightarrow \quad \boxed{\frac{d^2z}{d\theta^2} + z + \frac{Ze^2}{4\pi\varepsilon_0 M u^2 b^2} = 0}$$

Consider solutions of the form $\boxed{z(\theta) = A\cos\theta + B}$

$$\frac{dz}{d\theta} = -A\sin\theta \qquad \frac{d^2z}{d\theta^2} = -A\cos\theta = B - z$$

$$\therefore \quad B - z + z + \frac{Ze^2}{4\pi\varepsilon_0 M u^2 b^2} = 0$$

$$\Rightarrow \quad \boxed{B = \frac{-Ze^2}{4\pi\varepsilon_0 M u^2 b^2}}$$



Now velocity $\underline{v} = \dot{r}\,\hat{\underline{r}} + r\dot{\theta}\,\hat{\underline{\theta}}$

From above, $\boxed{\dot{r} = -ub\frac{dz}{d\theta}}$ and $\boxed{\dot{\theta} = \frac{ub}{r^2}}$

$$\therefore \quad \underline{v} = -ub\frac{dz}{d\theta}\hat{\underline{r}} + \frac{ub}{r}\hat{\underline{\theta}}$$

$$\therefore \quad \underline{v} = Aub\sin\theta\,\hat{\underline{r}} + ubz\,\hat{\underline{\theta}} \quad \leftarrow \boxed{\frac{1}{r} = z}$$

when $r \to \infty$, $z \to 0$, $\underline{v} \to \underline{u}$, $\theta \to \varepsilon$

$$\therefore \quad u = Aub\sin\varepsilon \quad \Rightarrow \quad \boxed{A = \frac{1}{b\sin\varepsilon}}$$

So $$0 = \frac{1}{b\sin\varepsilon}\cos\varepsilon - \frac{Ze^2}{4\pi\varepsilon_0 M u^2 b^2}$$ $[z = A\cos\theta + B]$ $[z \to 0,\ \theta \to \varepsilon]$

$$\Rightarrow \frac{Ze^2}{4\pi\varepsilon_0 Mu^2 b^2} = \frac{1}{b}\cot\varepsilon$$

$$\Rightarrow \quad b = \frac{Ze^2}{4\pi\varepsilon_0 Mu^2}\tan\varepsilon \quad \Rightarrow \tan\varepsilon = \frac{4\pi\varepsilon_0 Mu^2 b}{Ze^2}$$

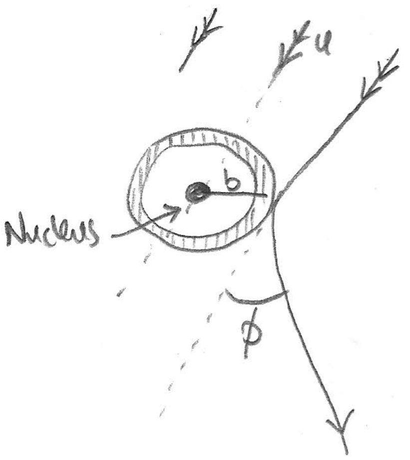
Now from geometry $\phi + 2\varepsilon = \pi \quad \therefore \varepsilon = \frac{\pi - \phi}{2}$

$$\therefore \tan\varepsilon = \frac{\tan\left(\frac{\pi}{2}\right) - \tan\frac{\phi}{2}}{1 + \tan\frac{\pi}{2}\tan\frac{\phi}{2}} = \frac{1 - \frac{\tan\phi/2}{\tan\frac{\pi}{2}}}{\frac{1}{\tan\frac{\pi}{2}} + \tan\phi/2}$$

Now $\tan\frac{\pi}{2} \to \infty$ So $\tan\varepsilon = \cot\frac{\phi}{2}$

$$\boxed{\cot\phi/2 = \frac{4\pi\varepsilon_0 Mu^2 b}{Ze^2}}$$

Now consider a beam of α particles incident upon a nucleus



The area of the beam cross section which corresponds to scattering within angular range ϕ to $\phi + d\phi$ is $\boxed{2\pi b\,db}$, since $b = b(\phi)$

$$\text{Now } \frac{d}{d\phi}\cot\phi/2 = \frac{d}{d\phi}\frac{1}{\tan\phi/2} = -\frac{1}{\tan^2\phi/2}\sec^2\frac{\phi}{2} \times \frac{1}{2}$$

$$= -\frac{1}{\frac{\sin^2\phi/2}{\cos^2\phi/2}} \times \frac{1}{\cos^2\phi/2}\frac{1}{2}$$

$$= \boxed{-\frac{1}{2}\frac{1}{\sin^2\frac{\phi}{2}}}$$

$$\therefore -\frac{1}{2}\frac{1}{\sin^2 \phi/2} = \frac{4\pi\varepsilon_0 M u^2}{Ze^2}\frac{db}{d\phi}$$

$$\Rightarrow \quad db = \frac{1}{2}\frac{-Ze^2}{4\pi\varepsilon_0 M u^2}\frac{d\phi}{\sin^2 \phi/2}$$

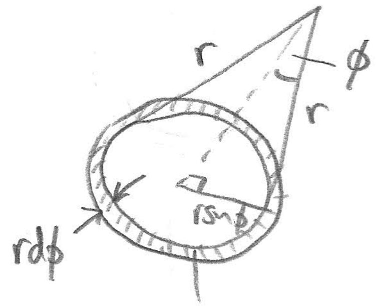
$$\therefore \quad 2\pi b\,db = 2\pi\left(\frac{Ze^2}{4\pi\varepsilon_0 M u^2}\right)\frac{\cos\phi/2}{\sin\phi/2}\,\frac{1}{2}\left(\frac{-Ze^2}{4\pi\varepsilon_0 M u^2}\right)\frac{d\phi}{\sin^2\phi/2}$$

$$\left[\, b = \frac{Ze^2 \frac{\cos\phi/2}{\sin\phi/2}}{4\pi\varepsilon_0 M u^2} \,\right] \qquad = -\pi\left(\frac{Ze^2}{4\pi\varepsilon_0 M u^2}\right)^2 \frac{\cos\phi/2}{\sin^3\phi/2}\,d\phi$$

Now α particles scattered by angle ϕ corresponds to a <u>Solid angle</u>

$$d\Omega = 2\pi\sin\phi\,d\phi$$

$$= \boxed{4\pi\sin\phi/2\cos\phi/2\,d\phi}$$



Area of sphere of radius r corresponding to Scattered α particles between ϕ and $\phi + d\phi$

is $2\pi r\sin\phi \times r\,d\phi$

$$\boxed{dA = 2\pi r^2 \sin\phi\,d\phi}$$

So <u>Solid angle</u> $d\Omega = \frac{dA}{r^2}$

$$= \boxed{2\pi\sin\phi\,d\phi}$$

Define <u>differential cross section</u> $\frac{d\sigma}{d\Omega}$ as the area of the incident beam which corresponds to a <u>unit Solid angle</u> of the Scattered α particles

let $d\sigma = |2\pi b\,db|$

$$\therefore \frac{d\sigma}{d\Omega} = \frac{\pi\left(\frac{Ze^2}{4\pi\varepsilon_0 M u^2}\right)^2 \frac{\cos\phi/2\,d\phi}{\sin^3\phi/2}}{4\pi\sin\phi/2\cos\phi/2\,d\phi}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{Ze^2}{8\pi\varepsilon_0 M u^2}\right)^2 \frac{1}{\sin^4\phi/2}}$$

ie $\propto \operatorname{cosec}^4 \phi/2$

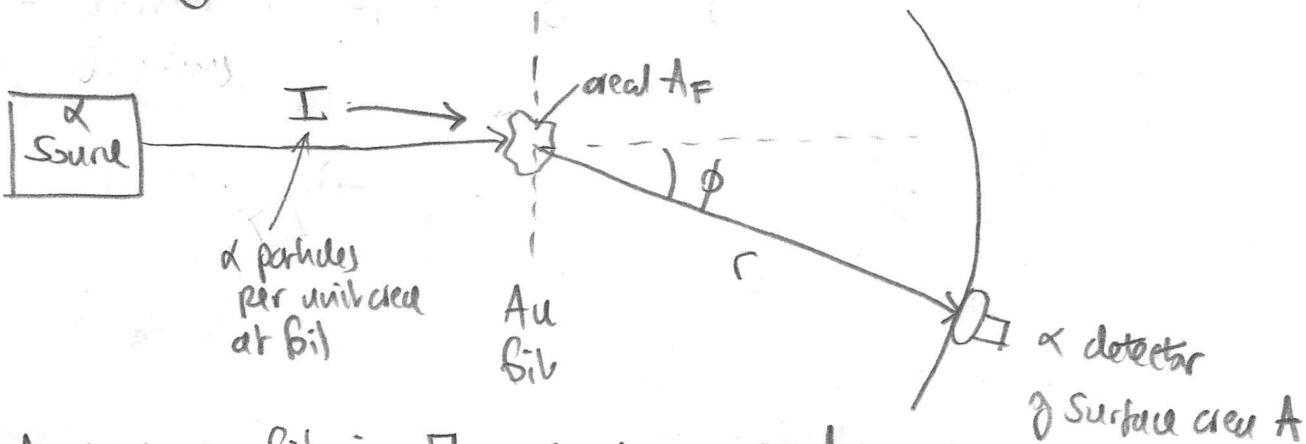
Note if I is the incident flux of α particles per unit area

$|2\pi b\,db|\,I$ would be the flux of α particles in scattering direction ϕ (well $\phi \to \phi + d\phi$)

$$\therefore \frac{d\sigma}{d\Omega} = \frac{|2\pi b\,db|}{d\Omega} = \frac{|2\pi b\,db|\,I}{d\Omega\;I}$$

$=$ flux of α particles in scattering direction ϕ (per unit solid angle) [← ie particles per second] / incident flux per unit area

This ratio could be measured experimentally, as in the classic by Rutherford, Geiger and Marsden in 1910-11.



Particle count at foil is Γ_0 counts per second $= I A_F$ (assume uniformly illuminated) This is measurable

Particle count at detector is $\Gamma = \frac{A}{r^2} \times \frac{d\sigma}{d\Omega} I \times N_{Au}$

($\frac{A}{r^2}$: solid angle of detector; $\frac{d\sigma}{d\Omega} I$: flux per unit solid angle; N_{Au}: # atoms in foil within beam)

N_{Au} = # atoms in foil of area A_F and thickness t_F $= n A_F t_F$ where n = number of atoms / unit volume.

So $\Gamma = \frac{A}{r^2} \times \left(\frac{ze^2}{8\pi\varepsilon_0 M u^2}\right)^2 \frac{1}{\sin^4 \phi/2} \frac{\Gamma_0}{\cancel{A_F}} \times n \cancel{A_F} t_F$

$$\Rightarrow \boxed{\frac{\Gamma}{\Gamma_0} = \frac{A t_F n}{r^2} \left(\frac{ze^2}{8\pi\varepsilon_0 M u^2}\right)^2 \frac{1}{\sin^4 \phi/2}}$$

For Au: $Z = 79$

$n \simeq 5.9 \times 10^{28}\ m^{-3}$

$A \sim 15\ mm^2 = 1.5 \times 10^{-5}\ m^2$

$t_F \sim 1.5\ \mu m = 1.5 \times 10^{-6}\ m$

Note $E_\alpha = \frac{1}{2} M u^2$ So $\boxed{\frac{\Gamma}{\Gamma_0} = \frac{A t_F n}{r^2}\left(\frac{ze^2}{4\pi\varepsilon_0 E}\right)^2 \frac{1}{\sin^4 \phi/2}}$

↑ kinetic energy

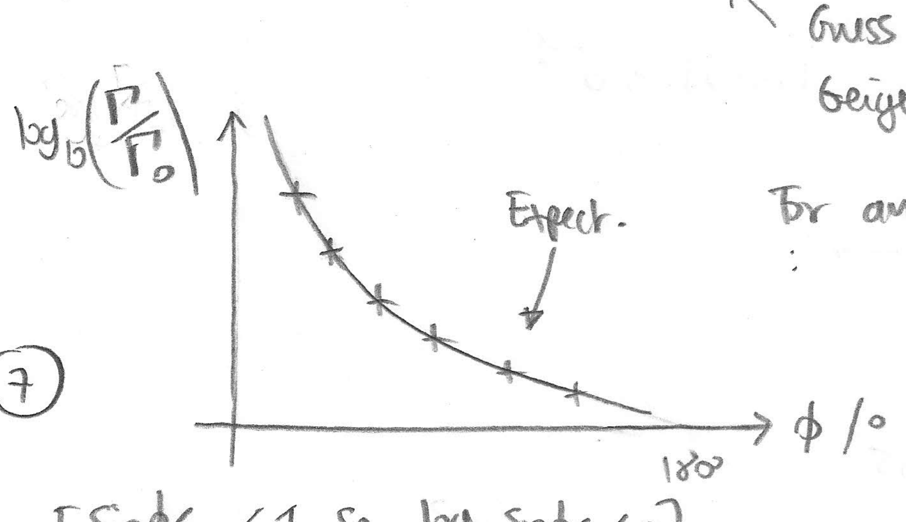
$$\therefore \boxed{\log_{10} \Gamma/\Gamma_0 = \log_{10}\left(\frac{A t_F n}{r^2}\left(\frac{ze^2}{4\pi\varepsilon_0 E}\right)^2\right) + 4 \log_{10}\left(\frac{1}{\sin \phi/2}\right)}$$

$\Gamma_0 \simeq 1000$ counts/min ← Typical for 'School Safe' Source.

$E_\alpha \sim 3\ MeV$

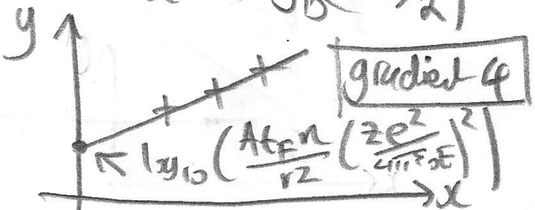
$e \sim 1.602 \times 10^{-19}\ C$ $\quad \varepsilon_0 = 8.85 \times 10^{-12}\ Fm^{-1}$

$r \sim 0.1\ m$ ← Guess at dimensions of Geiger & Marsden's apparatus



For analysis: $y = \log_{10}(\Gamma/\Gamma_0)$

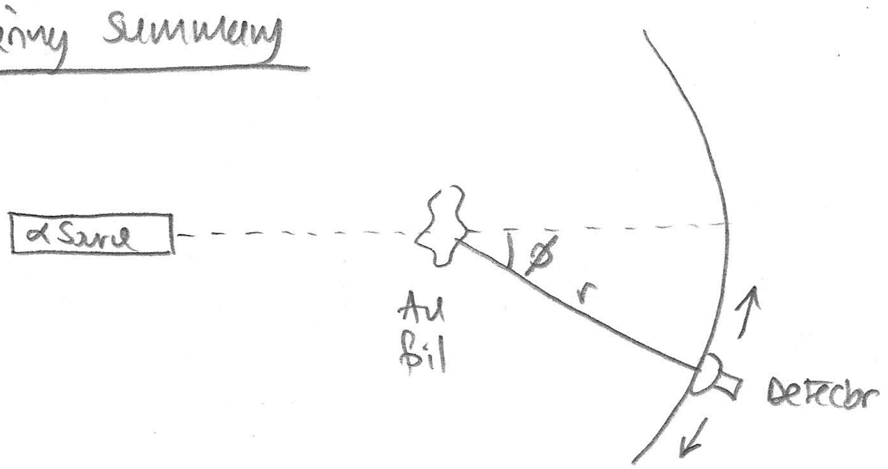
$x = \log_{10}\left(\frac{1}{\sin \phi/2}\right)$ ← +ve



[$\sin \phi/2 < 1$ so $\log_{10} \sin \phi/2 < 0$] ← Hence use $\frac{1}{\sin \phi/2}$ so log will be +ve.

Rutherford Scattering summary

Experiment:



$$\boxed{\frac{\Gamma}{\Gamma_0} = \frac{A t_F n}{r^2}\left(\frac{Ze^2}{4\pi\varepsilon_0 E}\right)^2 \frac{1}{\sin^4 \phi/2}}$$

Γ particle count rate at detector

Γ_0 " " " at foil

t_F foil thickness

n # atoms per m^3 in foil

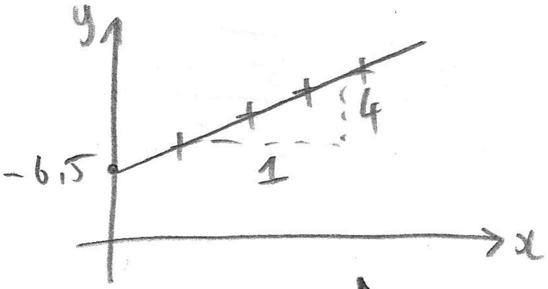
E α particle KE

Z atomic number (79 for gold)

e charge on electron

$$\underbrace{\log_{10}\frac{\Gamma}{\Gamma_0}}_{y} = \underbrace{\log_{10}\left[\frac{A t_F n}{r^2}\left(\frac{Ze^2}{4\pi\varepsilon_0 E}\right)^2\right]}_{c} + 4\underbrace{\log_{10}\left(\frac{1}{\sin \phi/2}\right)}_{x}$$

ie predict $\boxed{y = 4x + c}$



For sensible numbers:

$$\frac{A t_F n}{r^2}\left(\frac{Ze^2}{4\pi\varepsilon_0 E}\right)^2$$

$$\simeq 1.9101 \times 10^{-4}$$