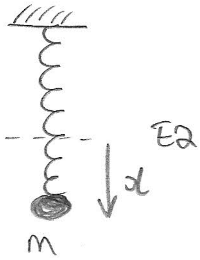
SIMPLE HARMONIC MOTION (SHM) AND SMALL OSCILLATIONS

1/ (i)



$$\boxed{\omega = \sqrt{\frac{k}{m}}}$$

Since from Newton II

$$m\ddot{x} = -kx$$

$$\therefore \ddot{x} = -\frac{k}{m}x$$

[Compare to SHM equation $\ddot{x} = -\omega^2 x$]

$$\omega = \frac{2\pi}{T} \quad \therefore \quad \frac{T}{2\pi} = \sqrt{\frac{M}{k}} \Rightarrow \boxed{T = 2\pi\sqrt{\frac{M}{k}}}$$

$$\text{so} \quad \left(\frac{T}{2\pi}\right)^2 = \frac{M}{k} \quad \therefore \quad \boxed{k = M\left(\frac{2\pi}{T}\right)^2}$$

{ Quick way: $k = m\omega^2$ }

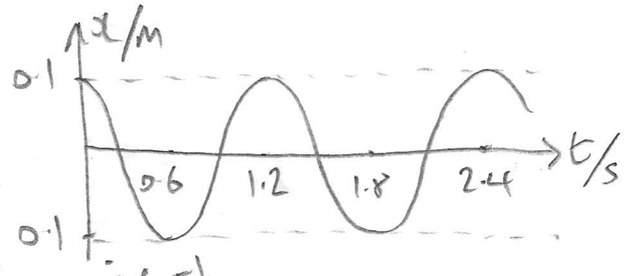
$$\text{So} \quad k = 0.2\,\text{kg} \times \left(\frac{2\pi}{1.2\,\text{s}}\right)^2 = \boxed{5.48\ \text{kg}\,\text{s}^{-2}}$$

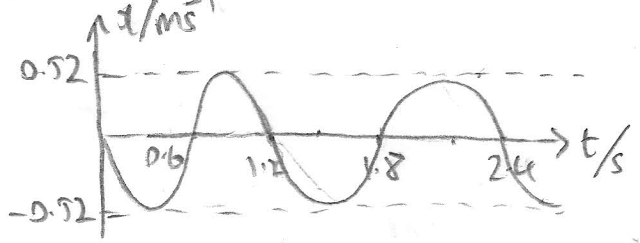
(ii)

$$\boxed{x = x_0 \cos(2\pi t/T)}$$

$x_0 = 0.1\,\text{m}$, $T = 1.2\,\text{s}$

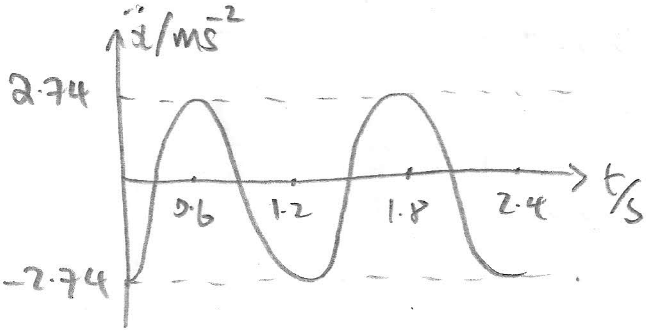
$$\boxed{\begin{aligned} \dot{x} &= -\frac{2\pi}{T} x_0 \sin(2\pi t/T) \\ \ddot{x} &= -\left(\frac{2\pi}{T}\right)^2 x_0 \cos(2\pi t/T) \end{aligned}}$$





$$\frac{2\pi}{T} x_0 = \frac{2\pi}{1.2} \times 0.1 = \boxed{0.52\,\text{m/s}}$$

$$\left(\frac{2\pi}{T}\right)^2 x_0 = \boxed{2.74\ \text{m/s}^2}$$



$$\dot{x}(2.0) = -\frac{2\pi}{1.2} \times 0.1 \sin\left(\frac{2\pi \times 2.0}{1.2}\right)$$

$$= \boxed{-0.45\ \text{m/s}}$$

(iii) $$KE = \tfrac{1}{2} m \dot{x}^2$$

$$= \tfrac{1}{2} m \left(-\frac{2\pi}{T} x_0 \sin\left(\frac{2\pi t}{T}\right)\right)^2$$

$$= \tfrac{1}{2} m \times \frac{4\pi^2}{T^2} x_0^2 \sin^2\left(\frac{2\pi t}{T}\right)$$

$$= \boxed{\frac{2\pi^2 m x_0^2}{T^2} \sin^2\left(\frac{2\pi t}{T}\right)}$$

So two maxima per period T.

$$EPE = \tfrac{1}{2} k x^2 = \tfrac{1}{2} k x_0^2 \cos^2\left(\frac{2\pi t}{T}\right)$$

Now from (i): $k = m\left(\frac{2\pi}{T}\right)^2$

So $$EPE = \tfrac{1}{2} m \times \frac{4\pi^2}{T^2} x_0^2 \cos^2\left(\frac{2\pi t}{T}\right)$$

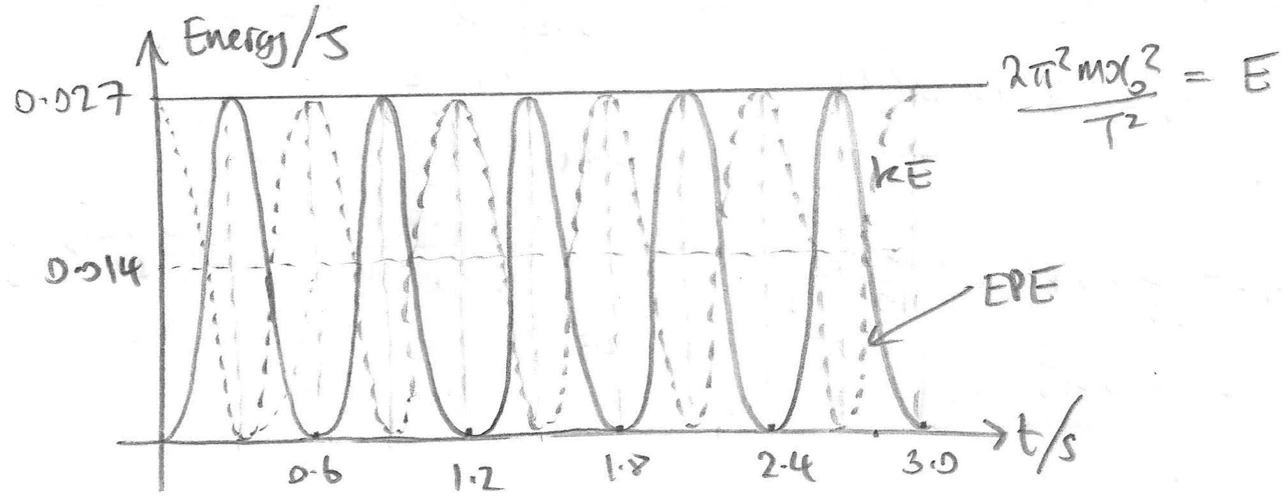
$$= \boxed{\frac{2\pi^2 m x_0^2}{T^2} \cos^2\left(\frac{2\pi t}{T}\right)}$$

Ditto.

Total energy $$E = KE + EPE = \frac{2\pi^2 m x_0^2}{T^2} \underbrace{\left[\cos^2\left(\frac{2\pi t}{T}\right) + \sin^2\left(\frac{2\pi t}{T}\right)\right]}_{=1}$$

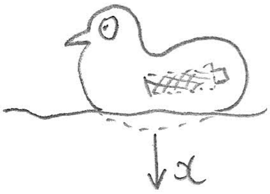
$$\Rightarrow \boxed{E = \frac{2\pi^2 m x_0^2}{T^2}}$$

ie $\boxed{\text{"}E = \tfrac{1}{2} m \omega^2 x_0^2\text{"}}$



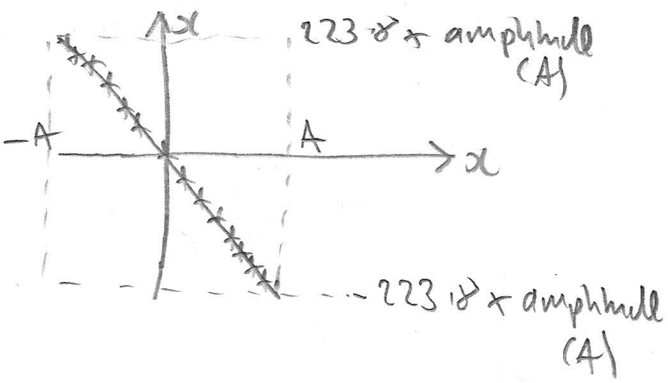
(2) Total energy $= \frac{2\pi^2 \times 0.2 \times 0.1^2}{1.2^2} = \boxed{0.027\text{ J}}$

(iv)



$$\ddot{x} = -\left(\frac{2\pi}{T}\right)^2 x$$

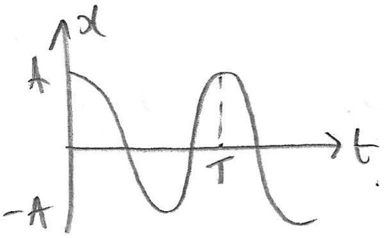
$$\boxed{\ddot{x} = -223.8\,x}$$



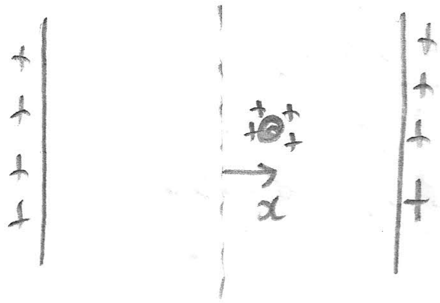
So $\left(\frac{2\pi}{T}\right)^2 = 223.8$

$\therefore \frac{2\pi}{T} = \sqrt{223.8}$ $\quad \therefore T = \frac{2\pi}{\sqrt{223.8}}$

$$\boxed{T = 0.42\text{s}}$$



(v)



$$\boxed{\begin{aligned} x &= A\cos(\omega t - \phi) \\ \dot{x} &= -A\omega\sin(\omega t - \phi) \\ \ddot{x} &= -\omega^2 x \end{aligned}}$$

when $t = 0\,s$, $x = 2.0$, $\dot{x} = -3.0$
and $\ddot{x} = -4$. (cm, cm/s, cm/s²).

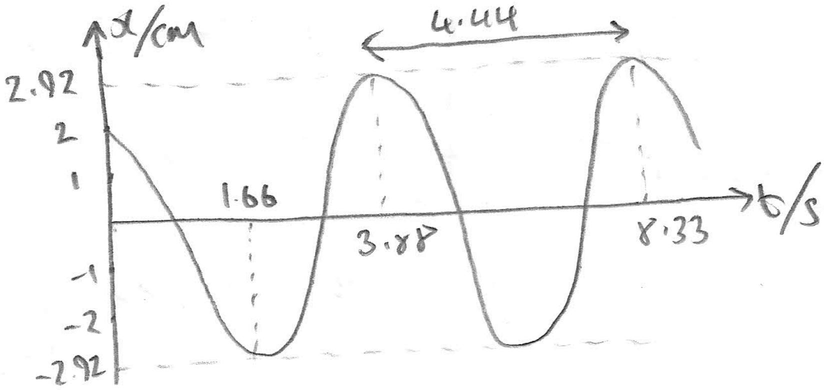
$\therefore$
$2 = A\cos(-\phi)$ ①
$-3 = -A\omega\sin(-\phi)$ ②
$-4 = -\omega^2 \times 2$ ③

so $\omega^2 = 2 \therefore \omega = \sqrt{2} \approx \boxed{1.41\ \text{rad}s^{-1}}$

$\frac{②}{①}: -\sqrt{2}\tan(-\phi) = -3/2 \Rightarrow \tan(-\phi) = \frac{3}{2\sqrt{2}}$

So $\phi = -\tan^{-1}\left(\frac{3}{2\sqrt{2}}\right) = \boxed{-0.815\ \text{radians}}$ $(-46.7°)$

In ①: $A = \frac{2}{\cos(-\phi)} = \boxed{2.92\ \text{cm}}$



$$\boxed{x(t) = 2.92\cos(1.41t + 0.815)}$$

when $t=0$, $x = 2.92\cos(0.815) \simeq 2.00$. (cm) ✓

Maxima when $1.41t + 0.815 = 2\pi n$

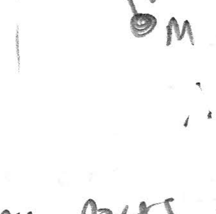
$$\Rightarrow t = \frac{2\pi n - 0.815}{1.41}$$

$t = 3.88s$, $8.33s$ ...

$n=1$ $\quad$ $n=2$

Note $\omega = \frac{2\pi}{T}$ $\therefore$ $T = \frac{2\pi}{\sqrt{2}} = \boxed{4.44s}$

(vi)



Eg a) Period $\boxed{T = k M^A g^B l^C}$

$[T] = s$ $\quad$ $[M] = kg$ $\quad$ $[g] = ms^{-2}$

$[l] = m$

$\therefore$ $s = k\, kg^A\, m^B s^{-2B}\, m^C$

comparing pairs of the SI quantities:

s: $\quad 1 = -2B$ $\quad\therefore$ $\boxed{B = -\frac{1}{2}}$

kg: $\quad 0 = A$ $\quad\therefore$ $\boxed{A = 0}$

m: $\quad 0 = B + C$ $\quad\therefore$ $C = -B$ $\quad\therefore$ $\boxed{C = \frac{1}{2}}$

So $T = k g^{-\frac{1}{2}} \ell^{\frac{1}{2}}$ or $\boxed{T = k\sqrt{\frac{\ell}{g}}}$

b) Mars church pendulum: $T = 2\pi\sqrt{\frac{3.14}{3.72}}$

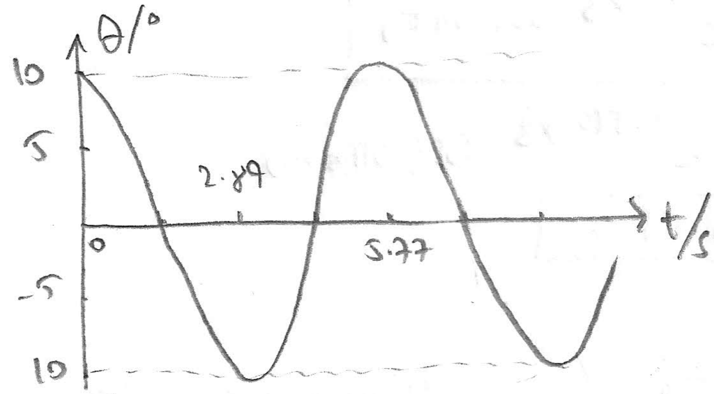
$= \boxed{5.77s}$

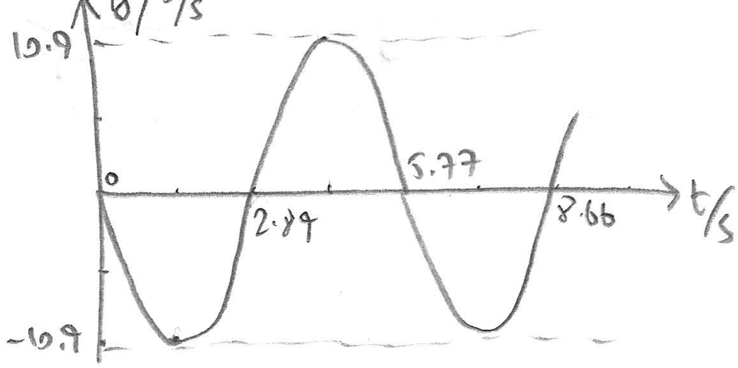
Assuming SHM since θ_{max} is only $10°$ (i.e. small angle approximation is $\simeq$ ok).

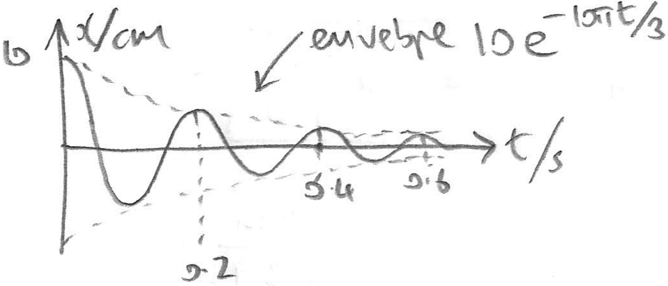
$$\boxed{\theta(t) = 10° \cos\left(\frac{2\pi t}{5.77}\right)}$$

$$\boxed{\dot{\theta}(t) = -\frac{2\pi}{5.77} \times 10° \sin\left(\frac{2\pi t}{5.77}\right)}$$

↳ i.e. $\dot{\theta}(t) = -10.9°/s \ \sin\left(\frac{2\pi t}{5.77}\right)$





(vii) 

If $f = 5$ Hz
period $T = 0.2$s

$$x(t) = 10\text{cm} \times e^{-\frac{1}{3}\omega t} \cos(2\pi t/T)$$

ie $$\boxed{x = 10e^{-10\pi t/3} \cos(10\pi t)}$$

$$\left[\omega = \frac{2\pi}{0.2} = 10\pi \text{ rad s}^{-1}\right]$$

when $t = 0.2$s, $\boxed{x = 1.23 \text{ cm}}$

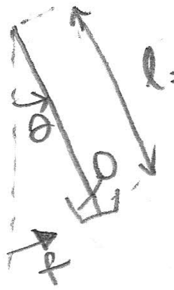
{ So a bit steeper than crude sketch above! }

$$\boxed{\dot{x} = -10e^{-10\pi t/3}(10\pi)\sin(10\pi t) - 10\left(\frac{10\pi}{3}\right)e^{-10\pi t/3}\cos(10\pi t)}$$

So: $$x(0.1) = 10e^{-10\pi(0.1)/3}\cos(10\pi \times 0.1)$$
$$= \boxed{-3.51 \text{ cm}}$$

$$\dot{x}(0.3) = -\frac{100\pi}{3}e^{-\frac{10\pi \times 0.3}{3}}\Big(3\underbrace{\sin(10\pi \times 0.3)}_{0} + \underbrace{\cos(10\pi \times 0.3)}_{-1}\Big)$$
$$= \boxed{4.53 \text{ cm/s}}$$

(viii)



$l = 4.2\,m$ $\qquad g = 9.81\, m/s^2$

$$\boxed{\ddot{\theta} + \frac{\pi}{6T}\dot{\theta} + \left(\frac{2\pi}{T}\right)^2 \theta = \left(\frac{2\pi}{T}\right)^2 \frac{\pi}{36} \sin \omega t}$$

describes the damped, driven SHM of the swing, driven by the force f which has a sinusoidal value, at frequency $\omega/2\pi$.

Compare with: $\boxed{\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = A_0\omega_0^2 \sin \omega t}$

so $\quad 2\gamma = \frac{\pi}{6T} \quad \therefore \gamma = \frac{\pi}{12T}$

$\omega_0^2 = \left(\frac{2\pi}{T}\right)^2 \quad \therefore \omega_0 = \frac{2\pi}{T} \qquad T = 2\pi\sqrt{\frac{l}{g}}$

$A_0 = \pi/36 \qquad \Rightarrow \boxed{\omega_0 = \sqrt{g/l}} \qquad$ Also $\gamma = \frac{2\pi}{T}\frac{1}{24}$

$\Rightarrow \boxed{\gamma = \frac{1}{24}\sqrt{\frac{g}{l}}}$

Steady state solution is $\boxed{\theta(t) = A\sin(\omega t - \phi)}$

where $A = \dfrac{A_0\omega_0^2}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2}}$ and $\phi = \tan^{-1}\left(\dfrac{2\gamma\omega}{\omega_0^2 - \omega^2}\right)$

ie/ $\boxed{A = \dfrac{\frac{\pi}{36}\frac{g}{l}}{\sqrt{\left(\frac{g}{l}-\omega^2\right)^2 + \frac{\omega^2}{144}\frac{g}{l}}}}$ and $\boxed{\phi = \tan^{-1}\left(\dfrac{\frac{1}{12}\sqrt{\frac{g}{l}}\,\omega}{\frac{g}{l} - \omega^2}\right)}$

$4\gamma^2 = \frac{4}{24^2}\frac{g}{l}$

$4/24^2 = \frac{1}{144}$

$g/l = \frac{9.81}{4.2} = 2.34. \qquad$ Also, $\omega = 2\pi f$.

Swing period is $T = 2\pi\sqrt{\frac{4.2}{9.81}} = \boxed{4.11\,s}$

Max amplitude of $\dfrac{A_0 \omega_0^2}{2\gamma\sqrt{\omega_0^2 - \gamma^2}}$

ie $$\frac{\frac{\pi}{36}\frac{g}{l}}{\frac{1}{12}\sqrt{\frac{g}{l}}\sqrt{\frac{g}{l} - \frac{g}{24^2 l}}} = \frac{\frac{\pi}{3}\frac{g}{l}}{\frac{g}{l}\sqrt{1-\frac{1}{24^2}}} = \frac{\pi}{3} \times \frac{1}{\sqrt{1-\frac{1}{24^2}}}$$

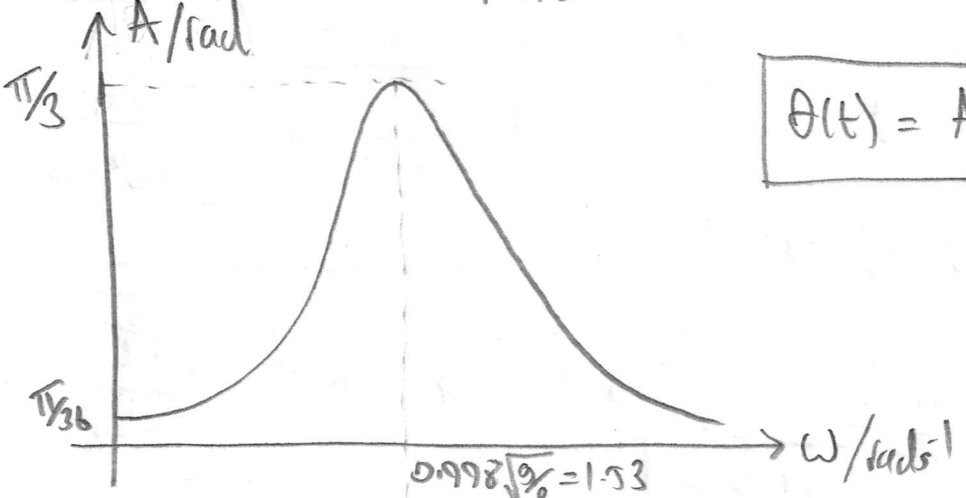
$$= \frac{\pi}{3} \times 1.00087$$

$$\simeq \boxed{\pi/3}$$

This occurs at $\omega = \sqrt{\frac{g}{l} - \frac{2g}{24^2 l}} = \sqrt{\frac{g}{l}}\left(1 - \frac{1}{2\gamma\gamma}\right)^{\frac{1}{2}}$

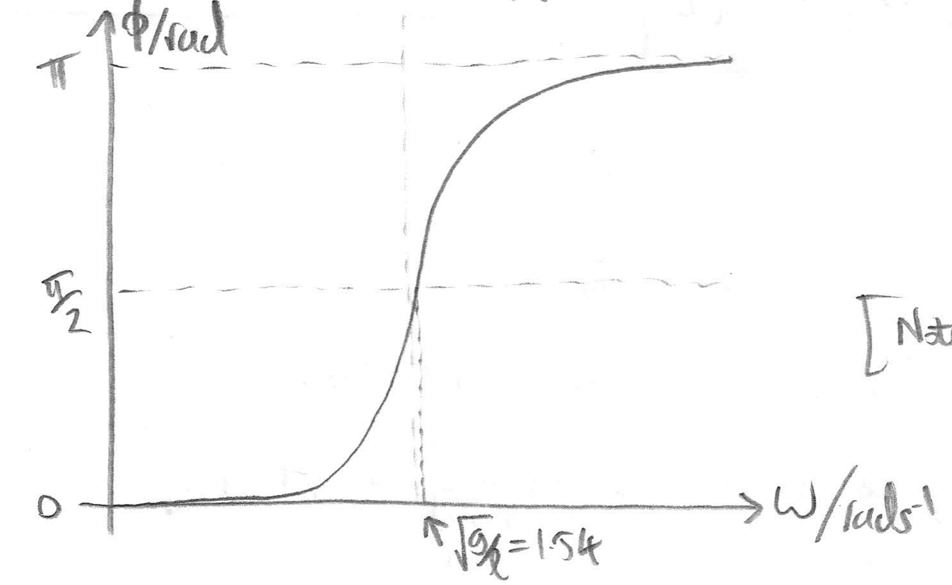
$$\simeq \boxed{0.998\sqrt{\frac{g}{l}}}$$

So $\omega_{max} = 0.998\sqrt{\frac{9.81}{4.2}} = \boxed{1.53 \text{ rads}^{-1}}$



$$\boxed{\theta(t) = A\sin(\omega t - \phi)}$$

"Amplitude response"



"phase response"

[Note $\phi = \pi/2$ when $\omega = \sqrt{g/l}$]

(ix) if the swing was critically damped $\boxed{\gamma = \omega_0}$

ie $\gamma = \sqrt{g/\ell}$

$$\therefore \boxed{\theta(t) = e^{-\gamma t}(A_1 + A_2 t)}$$

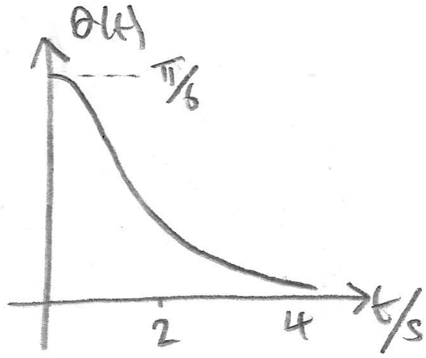
$$\theta(0) = \pi/6 \quad \therefore \quad \boxed{A_1 = \pi/6}$$

$$\dot\theta(t) = e^{-\gamma t}(A_2) + (A_1 + A_2 t)(-\gamma)e^{-\gamma t}$$

$$\dot\theta(0) = 0 \quad \text{so} \quad 0 = A_2 - \gamma \tfrac{\pi}{6}$$

$$\Rightarrow \quad \boxed{A_2 = \tfrac{\pi}{6}\sqrt{\tfrac{g}{\ell}}}$$

$$\therefore \boxed{\theta(t) = \tfrac{\pi}{6} e^{-t\sqrt{g/\ell}}\left(1 + t\sqrt{\tfrac{g}{\ell}}\right)}$$



(x) a) underdamped

$$\boxed{x(t) = Ae^{-\gamma t}\cos\left(t\sqrt{\omega^2-\gamma^2} - \phi\right)}$$

$$\dot x = \boxed{-\sqrt{\omega^2-\gamma^2}\,Ae^{-\gamma t}\sin\left(t\sqrt{\omega^2-\gamma^2} - \phi\right) - \gamma Ae^{-\gamma t}\cos\left(t\sqrt{\omega^2-\gamma^2} - \phi\right)}$$

$$\ddot x = \gamma\sqrt{\omega^2-\gamma^2}\,Ae^{-\gamma t}\sin\left(t\sqrt{\omega^2-\gamma^2} - \phi\right) - (\omega^2-\gamma^2)Ae^{-\gamma t}\cos\left(t\sqrt{\omega^2-\gamma^2} - \phi\right)$$

$$+\gamma^2 Ae^{-\gamma t}\cos\left(t\sqrt{\omega^2-\gamma^2} - \phi\right) + \gamma\sqrt{\omega^2-\gamma^2}\,Ae^{-\gamma t}\sin\left(t\sqrt{\omega^2-\gamma^2} - \phi\right)$$

$$= \boxed{2\gamma\sqrt{\omega^2-\gamma^2}\,Ae^{-\gamma t}\sin\left(t\sqrt{\omega^2-\gamma^2} - \phi\right) + (2\gamma^2 - \omega^2)Ae^{-\gamma t}\cos\left(t\sqrt{\omega^2-\gamma^2} - \phi\right)}$$

so $\ddot x + 2\gamma \dot x + \omega^2 x$

$$= Ae^{-\gamma t}\Big[\left(2\gamma\sqrt{\omega^2-\gamma^2} - 2\gamma\sqrt{\omega^2-\gamma^2}\right)\sin\left(t\sqrt{\omega^2-\gamma^2} - \phi\right)$$

$$+ \left(2\gamma^2 - \omega^2 - 2\gamma^2 + \omega^2\right)\cos\left(t\sqrt{\omega^2-\gamma^2} - \phi\right)\Big] = 0$$

✓

b) overdamped

$$\boxed{x(t) = e^{-\gamma t}\left(A_1 e^{t\sqrt{\gamma^2-\omega^2}} + A_2 e^{-t\sqrt{\gamma^2-\omega^2}}\right)}$$

$$\begin{aligned}\dot{x} &= -\gamma e^{-\gamma t}\left(A_1 e^{t\sqrt{\gamma^2-\omega^2}} + A_2 e^{-t\sqrt{\gamma^2-\omega^2}}\right) \\ &\quad + e^{-\gamma t}\sqrt{\gamma^2-\omega^2}\left(A_1 e^{t\sqrt{\gamma^2-\omega^2}} - A_2 e^{-t\sqrt{\gamma^2-\omega^2}}\right)\end{aligned}$$

$$\boxed{\dot{x} = e^{-\gamma t}\left(A_1\left(\sqrt{\gamma^2-\omega^2}-\gamma\right)e^{t\sqrt{\gamma^2-\omega^2}} - A_2\left(\sqrt{\gamma^2-\omega^2}+\gamma\right)e^{-t\sqrt{\gamma^2-\omega^2}}\right)}$$

$$\begin{aligned}\ddot{x} &= -\gamma e^{-\gamma t}\left(A_1\left(\sqrt{\gamma^2-\omega^2}-\gamma\right)e^{t\sqrt{\gamma^2-\omega^2}} - A_2\left(\sqrt{\gamma^2-\omega^2}+\gamma\right)e^{-t\sqrt{\gamma^2-\omega^2}}\right) \\ &\quad + e^{-\gamma t}\left(A_1\left(\gamma^2-\omega^2-\gamma\sqrt{\gamma^2-\omega^2}\right)e^{t\sqrt{\gamma^2-\omega^2}} + A_2\left(\gamma^2-\omega^2+\gamma\sqrt{\gamma^2-\omega^2}\right)e^{-t\sqrt{\gamma^2-\omega^2}}\right)\end{aligned}$$

$$\boxed{\begin{aligned}\ddot{x} = e^{-\gamma t}\Big(&A_1\left(-\gamma\sqrt{\gamma^2-\omega^2}+\gamma^2+\gamma^2-\omega^2-\gamma\sqrt{\gamma^2-\omega^2}\right)e^{t\sqrt{\gamma^2-\omega^2}} \\ &+ A_2\left(\gamma^2-\omega^2+\gamma\sqrt{\gamma^2-\omega^2}+\gamma\sqrt{\gamma^2-\omega^2}+\gamma^2\right)e^{-t\sqrt{\gamma^2-\omega^2}}\Big)\end{aligned}}$$

$\therefore\ \ddot{x} + 2\gamma\dot{x} + \omega^2 x =$

$$\begin{aligned}e^{-\gamma t}\Big(&A_1 e^{t\sqrt{\gamma^2-\omega^2}}\left\{2\gamma^2-\omega^2\underbrace{-2\gamma\sqrt{\gamma^2-\omega^2}+2\gamma\sqrt{\gamma^2-\omega^2}}_{=0}-2\gamma^2+\omega^2\right\} \\ &+ A_2 e^{-t\sqrt{\gamma^2-\omega^2}}\left\{2\gamma^2-\omega^2\underbrace{+2\gamma\sqrt{\gamma^2-\omega^2}-2\gamma\sqrt{\gamma^2-\omega^2}}_{=0}-2\gamma^2+\omega^2\right\}\Big)\end{aligned}$$

$= 0$ ✓

c) Critically damped $\quad x = e^{-\gamma t}(A_1 + A_2 t)$

$$\dot{x} = -\gamma e^{-\gamma t}(A_1 + A_2 t) \quad + e^{-\gamma t} A_2$$

$$\ddot{x} = \gamma^2 e^{-\gamma t}(A_1 + A_2 t) - \gamma e^{-\gamma t} A_2 \quad - \gamma e^{-\gamma t} A_2$$

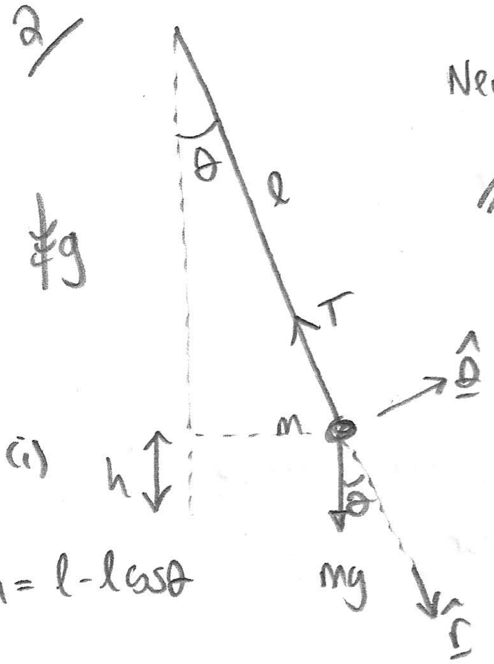
$$\therefore \ddot{x} + 2\gamma \dot{x} + \omega^2 x$$

$$= e^{-\gamma t}\left\{ A_1\left(\gamma^2 - 2\gamma^2 + \omega^2\right) + A_2\left(\gamma^2 t - \gamma - \gamma - 2\gamma^2 t + 2\gamma + \omega^2 t\right) \right\}$$

Now for critical damping $\boxed{\omega = \gamma}$

$$\therefore \ddot{x} + 2\gamma\dot{x} + \omega^2 x = e^{-\gamma t}\left\{ A_1 \left(\underbrace{\gamma^2 - 2\gamma^2 + \gamma^2}_{=0}\right) + A_2\left(\gamma^2 t \underbrace{- 2\gamma - 2\gamma^2 t + 2\gamma + \gamma^2 t}_{=0}\right)\right\} = 0 \checkmark$$

2/



$h = l - l\cos\theta$

Newton II for pendulum bob of mass m

$/\!/\hat{r}: \quad -ml\dot{\theta}^2 = -T + mg\cos\theta \quad$ radial

$/\!/\hat{\theta}: \quad ml\ddot{\theta} = -mg\sin\theta \quad$ tangential

if $\sin\theta \simeq \theta \quad$ (i.e. $\theta \ll 1$ radian) ideally

$$\boxed{\ddot{\theta} \simeq -\frac{g}{l}\theta} \qquad \text{SHM} \quad \ddot{\theta} = -\omega^2\theta$$

So $\omega = \sqrt{\frac{g}{l}} \quad \therefore \frac{2\pi}{\tau} = \sqrt{\frac{g}{l}} \quad \therefore \boxed{\tau = 2\pi\sqrt{\frac{l}{g}}}$

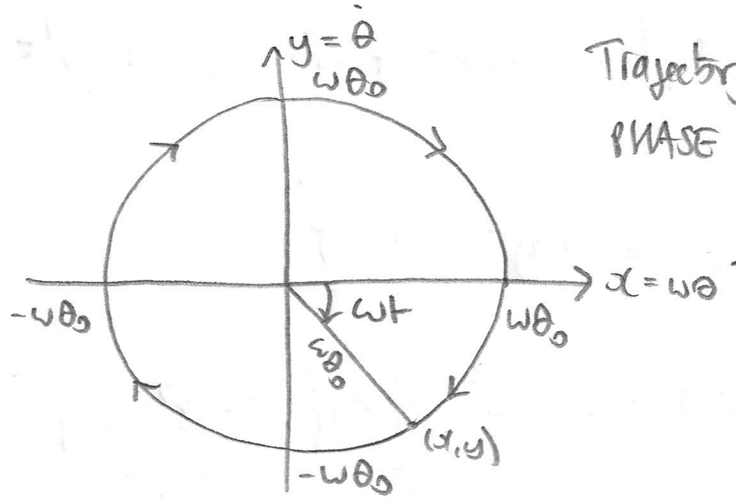
period

(ii) If motion were SHM $\boxed{\theta = \theta_0 \cos\omega t}$ [Released from θ_0 at rest]

where $\boxed{\omega = \sqrt{g/l}}$ $\boxed{\dot\theta = -\omega\theta_0 \sin\omega t}$

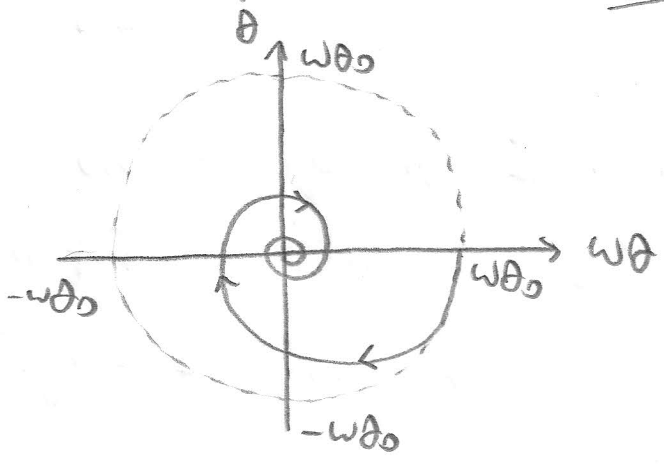
So since $\frac{2\pi}{T} = \omega$; let $x = \omega\theta$, $y = \dot\theta$

$x = \omega\theta_0 \cos\omega t$; $y = -\omega\theta_0 \sin\omega t$



Trajectory of $\dot\theta$ vs $\omega\theta$.
PHASE SPACE is a circle

If there is damping, trajectory will inspiral.



[Note damping may not be $\propto$ velocity, but perhaps $|\text{velocity}|^2$ if the pendulum is moving fast]

(iii) Conservation of energy for (undamped) pendulum

$$mgh_0 = mgh + \tfrac{1}{2}mv^2 \qquad h_0 = l(1-\cos\theta_0)$$

$$\therefore \quad v = \sqrt{2g(h_0 - h)}$$

Now from (i) $h = l(1-\cos\theta)$

$$\therefore \quad v = \sqrt{2gl}\left(1-\cos\theta_0 - 1 + \cos\theta\right)^{\frac{1}{2}}$$

$$\boxed{v = \sqrt{2gl}\left(\cos\theta - \cos\theta_0\right)^{\frac{1}{2}}}$$

This is speed (multiply by sign of $\dot{\theta}$ to get $\dot{\theta}/l$)

Now $\boxed{v = l\dot{\theta}}$ (is purely tangential)

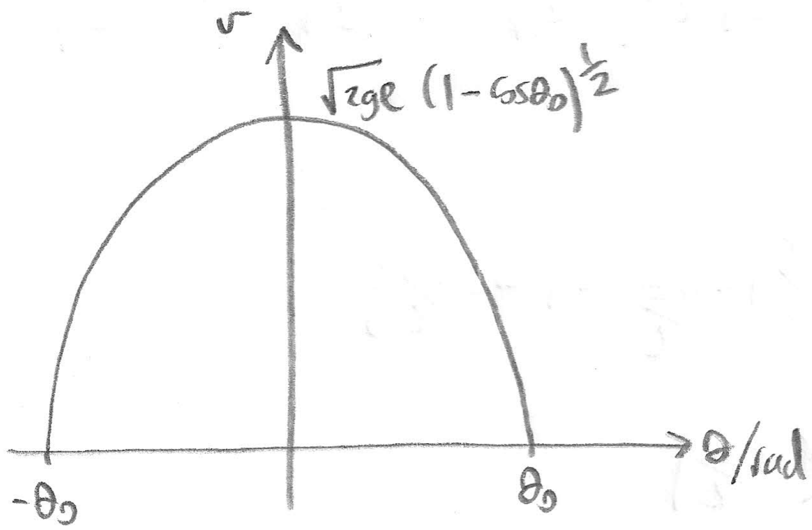
so from radial NII: $-\dfrac{mv^2}{l} = -T + mg\cos\theta$

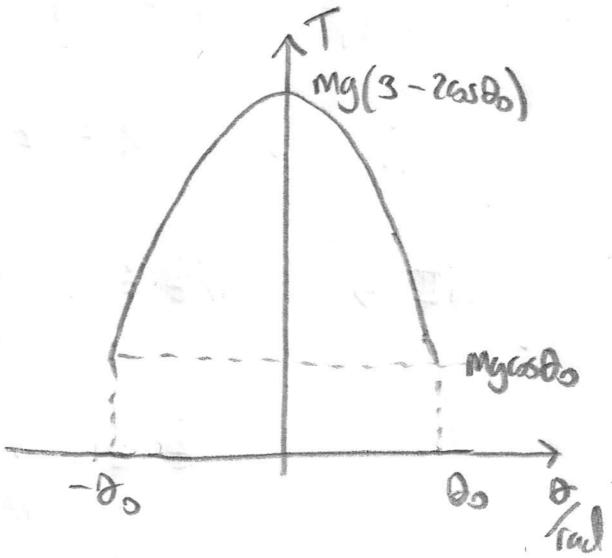
$$\Rightarrow \quad T = mg\cos\theta + mv^2/l$$

$$T = mg\cos\theta + 2mg(\cos\theta - \cos\theta_0)$$

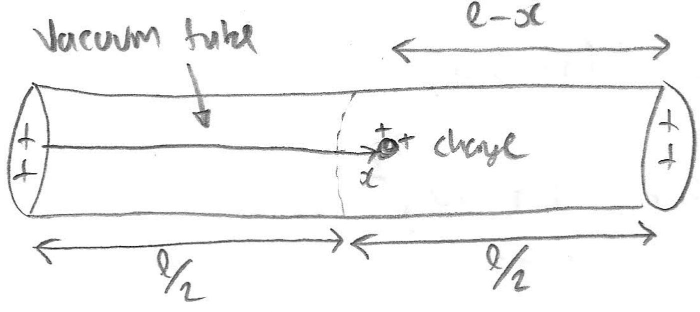
$$T = 3mg\cos\theta - 2mg\cos\theta_0$$

$$\boxed{T = mg(3\cos\theta - 2\cos\theta_0)}$$

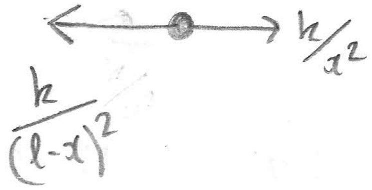




3/



Forces on charge are
ie repulsion from charged plates.



(i) So by Newton II $\rightarrow$ +ve :

$$\boxed{m\ddot{x} = \frac{k}{x^2} - \frac{k}{(l-x)^2}}$$

(ii) let $x = \frac{1}{2}l + z$ where $|z| << l$

$$\frac{1}{x^2} = \left(\tfrac{1}{2}l + z\right)^{-2} = \left(\frac{l}{2}\right)^{-2}\left(1 + \frac{2z}{l}\right)^{-2}$$

$$\simeq \left(\frac{l}{2}\right)^{-2}\left(1 - \frac{4z}{l}\right)$$

$$\frac{1}{(l-x)^2} = \left(\tfrac{1}{2}l - z\right)^{-2} = \left(\frac{l}{2}\right)^{2}\left(1 - \frac{2z}{l}\right)^{-2} \simeq \left(\frac{l}{2}\right)^{-2}\left(1 + \frac{4z}{l}\right)$$

$\therefore$ since $\ddot{x} = \ddot{z}$:

$$\text{N II} \Rightarrow \quad m\ddot{z} \simeq k\left(\frac{2}{l}\right)^2 \left\{1 - \frac{4z}{l} - 1 - 4z/l\right\}$$

$$\ddot{z} \simeq -\frac{4k}{Ml^2}\left(8\frac{z}{l}\right)$$

$$\therefore \boxed{\ddot{z} \simeq -\frac{32k}{Ml^3} z}$$

ie compared to SHM $\ddot{z} = -\omega^2 z$

$$\Rightarrow \quad \omega = \sqrt{\frac{32k}{Ml^3}} \qquad \omega = 2\pi f$$

So frequency of oscillations $f = \frac{\omega}{2\pi}$

$$\therefore \boxed{f = \frac{1}{2\pi}\sqrt{\frac{32k}{m\ell^3}}}$$

(iii) let $k = \frac{e^2}{4\pi\varepsilon_0}$ (Coulomb's law)

So $$f = \left(\frac{32e^2}{4\pi\varepsilon_0}\,\frac{1}{4\pi^2 m\ell^3}\right)^{1/2}$$

$$\boxed{f = \left(\frac{2e^2}{\pi^3\varepsilon_0 m\ell^3}\right)^{1/2}}$$

let $e = 1.602\times10^{-19}\,C$ (|charge| of electron or proton)
$m = 1.67\times10^{-27}\,kg$ (proton mass)
$\varepsilon_0 = 8.85\times10^{-12}\,Nm^2C^{-2}$ (permittivity of free space)
$\ell = 1.23\times10^{-6}\,m$ (molecular bond length)

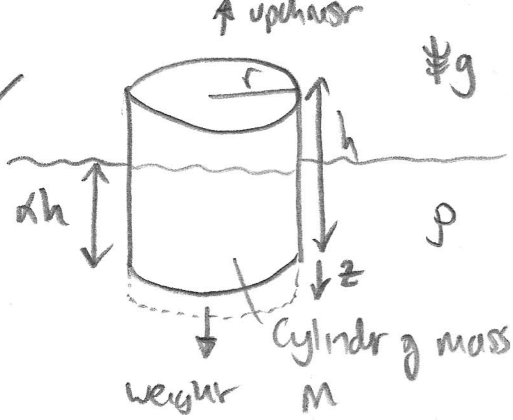
$\Rightarrow f \approx 2.45\times10^{14}\,Hz$

i.e. EM wavelength $\lambda \approx \frac{2.998\times10^8}{2.45\times10^{14}} \simeq \boxed{1222\,nm}$

which is infra red. Could easily be optical if ℓ was a bit smaller.

[400 – 650 nm]

4/



Archimedes: upthrust = weight of fluid displaced.

so in eq: $Mg = \rho\pi r^2 \alpha h g$

$\therefore \boxed{M = \rho \pi r^2 \alpha h}$

(So mass of displaced fluid = mass of cylinder in eq).

If cylinder is displaced by extra depth βh $(\beta \ll \alpha)$

Newton II: $M\ddot{z} = -\rho\pi r^2 z g$

$\ddot{z} = -\frac{\rho\pi r^2 g}{M} z$ SHM

ie $\omega^2 = \frac{\rho \pi r^2 g}{M}$

so $\frac{2\pi}{T} = \sqrt{\frac{\rho\pi r^2 g}{M}} \Rightarrow \boxed{T = 2\pi\sqrt{\frac{M}{\rho\pi r^2 g}}}$

Now $M = \rho\pi r^2 \alpha h$

So $T = 2\pi\sqrt{\frac{\cancel{\rho\pi r^2}\alpha h}{\cancel{\rho\pi r^2} g}}$

$$\boxed{T = 2\pi\sqrt{\frac{\alpha h}{g}}}$$

For iceberg: $r = 100\text{m}$, $h = 200\text{m}$. $M = \rho_{ice}\pi r^2 h$

So $\underbrace{\rho_{ice}\pi r^2 h}_{M} = \rho_w \pi r^2 \alpha h \Rightarrow \alpha = \frac{\rho_{ice}}{\rho_w} = \frac{920}{1025}$

$\simeq \boxed{0.899}$

* So an ice berg is $\simeq$ 90% Submerged! * $\rightarrow$ Titanic...

$$\therefore \quad T = 2\pi\sqrt{\frac{920}{1025} \times \frac{200}{9.81}}$$

$$\simeq \boxed{26.9\,s}$$

5/ $$\boxed{\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = A_0\omega_0^2 e^{i\omega t}}$$ ("complex SHM")

let $z = Ae^{i(\omega t - \phi)}$

$\dot{z} = i\omega z \; ; \; \ddot{z} = -\omega^2 z$ { MUCH easier than using sin and cos }

So $-\omega^2 z + 2i\gamma\omega z + \omega_0^2 z = A_0\omega_0^2 e^{i\omega t}$

$$Ae^{i\omega t}e^{-i\phi}\left(-\omega^2 + 2i\gamma\omega + \omega_0^2\right) = A_0\omega_0^2 e^{i\omega t}$$

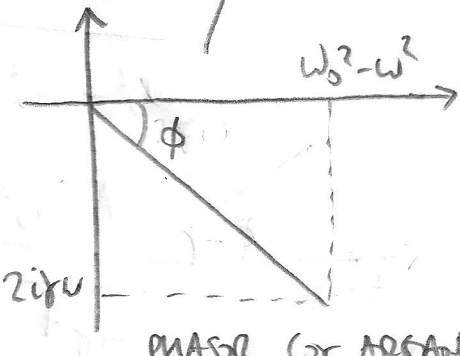
So $$Ae^{-i\phi} = \frac{A_0\omega_0^2}{\omega_0^2 - \omega^2 + 2i\gamma\omega}$$

$$Ae^{-i\phi} = \frac{A_0\omega_0^2\left(\omega_0^2 - \omega^2 - 2i\gamma\omega\right)}{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2}$$

$$\left[\frac{1}{a+ib} = \frac{a-ib}{(a+ib)(a-ib)} = \frac{a-ib}{a^2 - i^2b^2} = \frac{a-ib}{a^2+b^2}\right]$$

If A is real, then since $|e^{-i\phi}| = 1$
(and >0)

$$A = \frac{A_0\omega_0^2}{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2}\sqrt{\left(\omega_0^2-\omega^2\right)^2 + 4\gamma^2\omega^2}$$



PHASOR (or ARGAND) diagram

$$\therefore \boxed{A = \frac{A_0 \omega_0^2}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2}}}$$

Now from Argand / phasor diagram

$$\tan\phi = \frac{2\gamma\omega}{\omega_0^2-\omega^2}$$

so $\boxed{\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2-\omega^2}\right)}$

Now maximum A is when $\frac{dA}{d\omega} = 0$

$$\frac{dA}{d\omega} = -\frac{1}{2}\left((\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2\right)^{-3/2} A_0\omega_0^2 \left(2(\omega_0^2-\omega^2)(-2\omega) + 8\gamma^2\omega\right)$$

so $\frac{dA}{d\omega} = 0$ when $8\gamma^2\omega = 2(\omega_0^2-\omega^2)(2\omega)$

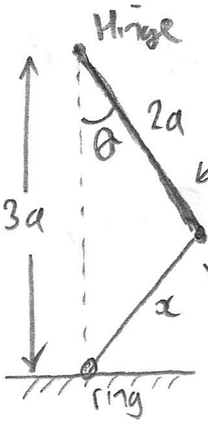
$$2\gamma^2 = \omega_0^2 - \omega^2$$

$$\therefore \boxed{\omega = \sqrt{\omega_0^2 - 2\gamma^2}}$$

$$A_{max} = \frac{A_0\omega_0^2}{\sqrt{(\omega_0^2 - \omega_0^2 + 2\gamma^2)^2 + 4\gamma^2(\omega_0^2 - 2\gamma^2)}}$$

$$= \frac{A_0\omega_0^2}{\sqrt{4\gamma^4 + 4\gamma^2\omega_0^2 - 8\gamma^4}} = \boxed{\frac{A\omega_0^2}{2\gamma\sqrt{\omega_0^2 - 2\gamma^2}}}$$

6/ $\downarrow g$

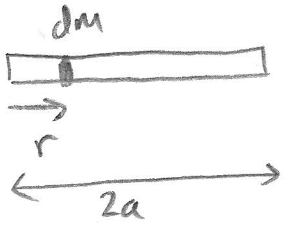


rod of mass m and length 2a

Threaded elastic of extension x, $k = 5mg/2a$

(i) Moment of inertia of rod rotating about the hinge is

$$I = \int_0^{2a} dm \times r^2$$



$$dm = \frac{dr}{2a} m$$

$$\therefore \quad I = \int_0^{2a} \frac{m}{2a} r^2 dr$$

$$I = \frac{m}{2a}\left[\frac{1}{3}r^3\right]_0^{2a} = \frac{m}{2a}\frac{1}{3}(8a^3)$$

$$= \boxed{\frac{4}{3}ma^2}$$

(ii) Cosine rule: $x^2 = (3a)^2 + (2a)^2 - 2(2a)(3a)\cos\theta$

$$x^2 = 9a^2 + 4a^2 - 12a^2\cos\theta$$

$$x^2 = 13a^2 - 12a^2\cos\theta$$

$$\boxed{x = a\sqrt{13 - 12\cos\theta}}$$

(iii) Total energy of the system is:

$$E = \underbrace{\frac{1}{2}I\dot{\theta}^2}_{KE} + \underbrace{mga(1-\cos\theta)}_{GPE} + \overbrace{\frac{1}{2}\frac{5mg}{2a}x^2}^{EPE}$$

(Height of CoM of rod is $3a - a\cos\theta$. At eq it is $2a$, so GPE gain is $a - a\cos\theta$)

$\therefore \quad E = \frac{1}{2}\left(\frac{4}{3}ma^2\right)\dot\theta^2 + mga(1-\cos\theta) + \frac{5}{4}\frac{Mg}{a}\left(13a^2 - 12a^2\cos\theta\right)$

$E = \frac{2}{3}ma^2\dot\theta^2 + mga\left(1 - \cos\theta + 16\frac{1}{4} - 15\cos\theta\right)$

$$\boxed{E = \frac{2}{3}ma^2\dot\theta^2 + mga\left(17\tfrac{1}{4} - 16\cos\theta\right)}$$

(iv) $\dot E = \frac{4}{3}ma^2\dot\theta\ddot\theta + 16mga\sin\theta\,\dot\theta$

$\dot E = ma\dot\theta\left(\frac{4}{3}a\ddot\theta + 16g\sin\theta\right)$

$\dot E = \frac{4}{3}ma^2\dot\theta\left(\ddot\theta + \frac{12g}{a}\sin\theta\right)$

if $\dot E = 0 \Rightarrow \boxed{\ddot\theta = -\frac{12g}{a}\sin\theta}$ (since $\dot\theta \neq 0$ $\forall\theta$)

if $\theta \ll 1$ (ie small angles)

$\ddot\theta \simeq -\frac{12g}{a}\theta$ ie SHM

$\omega = \sqrt{\frac{12g}{a}}$ so period $\boxed{T = 2\pi\sqrt{\frac{a}{12g}}}$

(v) if no elastic: $E = \frac{2}{3}ma^2\dot\theta^2 + mga(1-\cos\theta)$

(ie ignore $\frac{1}{2}kx^2$ term). $\therefore \dot E = \frac{4}{3}ma^2\dot\theta\ddot\theta + mga\sin\theta\dot\theta$

$\dot E = 0$ and $\sin\theta \simeq \theta \Rightarrow$ $\left(\ddot\theta\frac{4}{3}ma^2 + mga\theta\right)\dot\theta \simeq 0$

$\Rightarrow \boxed{\ddot\theta \simeq \frac{3g}{4a}\theta}$ ie SHM with period $\boxed{T' = 2\pi\sqrt{\frac{4a}{3g}}}$

so $T'/T = \sqrt{\frac{4}{3}}\Big/\sqrt{\frac{1}{12}} = \sqrt{\frac{4 \times 12}{3}} = \boxed{4}$