SHM differential equation: $\ddot{x} = -\omega^2 x$ with solution $x(t) = A\cos(\omega t - \phi)$. i.e. an oscillation of frequency $f = \frac{1}{2\pi}\omega$ and period $T = 1/f$. i.e. $\omega = 2\pi/T$. A is the amplitude and ϕ is a constant phase shift. Note $\ddot{x} = d^2x/dt^2$, $\dot{x} = dx/dt$.

Also: $\ddot{x} = \dot{x}\dfrac{d\dot{x}}{dx}$. So $\dot{x}\dfrac{d\dot{x}}{dx} = -\omega^2 x \Rightarrow \frac{1}{2}\dot{x}^2 = -\frac{1}{2}\omega^2 x^2 + c \Rightarrow \dot{x}^2 = \omega^2\left(A^2 - x^2\right)$. i.e. $\dot{x} = \pm\omega\sqrt{A^2 - x^2}$

Note also that: $\dot{x}(t) = -\omega A\sin(\omega t - \phi)$ and $A^2\sin^2(\omega t - \phi) = A^2 - A^2\cos^2(\omega t - \phi) = A^2 - x^2$.

Example: mass on a spring (where x is the displacement *from* equilibrium)

Newton II: $m\ddot{x} = -kx$ $\therefore \ddot{x} = -\dfrac{k}{m}x$. So SHM with $\omega = \sqrt{\dfrac{k}{m}}$.

Energy conservation: $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \Rightarrow m\dot{x}\ddot{x} + kx\dot{x} = 0 \Rightarrow \dot{x}\left(\ddot{x} + \dfrac{k}{m}x\right) = 0$ $\therefore \ddot{x} = -\dfrac{k}{m}x$.

Damped SHM: $\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0$. e.g. mass on a spring with damping force proportional to velocity, but acts in opposition to movement. Newton II: $m\ddot{x} = -kx - 2\gamma m\dot{x}$. Note each has *two* constants to be found from $x(0)$ and $\dot{x}(0)$.

If $\omega > \gamma$: 'underdamped' and solutions are oscillatory. $x(t) = A_1 e^{-\gamma t}\cos\left(t\sqrt{\omega^2 - \gamma^2} - \Phi\right)$

If $\omega < \gamma$: 'overdamped' and solutions don't oscillate. $x(t) = e^{-\gamma t}\left(A_1 e^{t\sqrt{\gamma^2 - \omega^2}} + A_2 e^{-t\sqrt{\gamma^2 - \omega^2}}\right)$

If $\omega = \gamma$: 'critically damped' and solutions don't quite oscillate. $x(t) = e^{-\gamma t}\left(A_1 + A_2 t\right)$

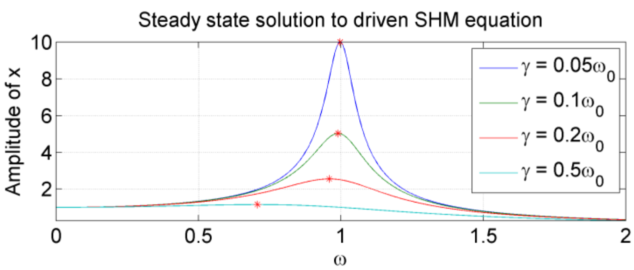
The time variation of these exponentially decaying terms is known as the *transient*, or 'impulse response.'

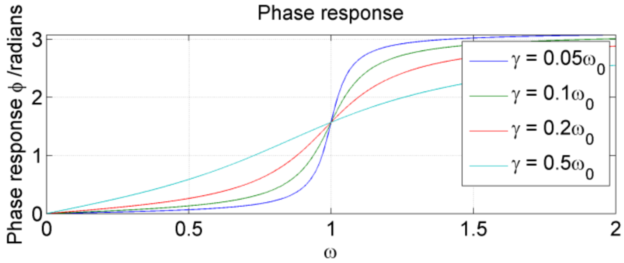
Driven SHM and resonance: $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = A_0\omega_0^2\sin\omega t$.

If *underdamped*, **steady state solution** (i.e. when the transient decays sufficiently to be neglected i.e. when $\gamma t \gg 1$) is:

$$x(t) = A\sin(\omega t - \phi). \qquad A(\omega) = A_0\omega_0^2 \Big/ \sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2}. \qquad \phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2 - \omega^2}\right).$$

Maximum oscillation amplitude $A_{max} = \dfrac{A_0\omega_0^2}{2\gamma\sqrt{\omega_0^2 - \gamma^2}}$ when $\omega = \sqrt{\omega_0^2 - 2\gamma^2}$.





Question 1

(i) A mass of 0.2kg undergoes oscillations of period 1.2s on a vertically mounted spring. Determine the spring constant k.

(ii) If the mass-spring system in (i) is initially stretched by 0.1m from equilibrium and then released, calculate the velocity of the mass after 2.0s. Sketch $x(t), \dot{x}(t), \ddot{x}(t)$ on separate graphs with the same time scale.

(iii) Determine expressions for the time variation of (a) kinetic energy and (b) potential energy for the mass-spring system in (i),(ii). On the same graph, sketch kinetic energy, potential energy and total energy vs time for at least two periods.

(iv) A rubber duck is placed in a bath. The duck has an in-built accelerometer with a USB stick memory for data logging. Analysis of the bobbing motion of the duck results in a straight line correlation between acceleration $\ddot{x}$ (in cm/s^2) and displacement x (in cm) of the form $\ddot{x} = -223.8x$. Determine the period of the oscillation in s.

(v) A charged ping-pong ball can be made to oscillate between two metal plates, both carrying the same charge with the same sign as the charge on the ping-pong ball. The ball has a displacement x of 2.0cm, a velocity of -3.0cm/s and an acceleration of -4.0cm/s at $t = 0.0\text{s}$. Determine A, ω, ϕ and hence sketch $x(t)$ for two periods.

(vi) A long pendulum of length l undergoes small oscillations with period $T = km^A g^B l^C$. m is the mass of the pendulum bob and g is the strength of gravity. k is a dimensionless constant.

(a) Use dimensional analysis to find A, B, C and hence show that $T = k\sqrt{l/g}$.

(b) As you will prove later (Q2), $k = 2\pi$. The First Church of Mars conducts commences Wednesday services with a swing of a 3.14m pendulum, starting from rest at a maximum angle of 10^{o}. Calculate the period of the pendulum, and its maximum speed. Hence sketch $\theta(t)$ and $\dot{\theta}(t)$. $g_{mars} = 3.72\text{m/s}^2$.

(vii) The suspension in an old car is *underdamped*. When the car hits a bump, it oscillates with frequency 5Hz. A particular bump causes a oscillation of initial amplitude 10cm, starting from rest. Assuming the damping constant $\gamma = \frac{1}{10}\omega$, determine an equation for the subsequent motion, and sketch it. What is the amplitude and velocity after 0.15s? Note: $x(t) = Ae^{-\gamma t}\cos\left(t\sqrt{\omega^2 - \gamma^2} - \phi\right)$ and $\omega = 2\pi f \times \left(1 - \frac{\gamma^2}{\omega^2}\right)^{-\frac{1}{2}}$.

(viii) A swing at a children's playground for danger-seeking nieces of Physicists is $l =$ 4.2m long. It can be modeled as a simple pendulum of period $T = 2\pi\sqrt{l/g}$ where $g = 9.81\text{m/s}^2$. The swing is driven by a sinusoidal driving force of frequency $f = \frac{1}{2\pi}\omega$, such that he (angular) motion of the swing is described by the differential equation $\ddot{\theta} + \frac{\pi}{6T}\dot{\theta} + \left(\frac{2\pi}{T}\right)^2\theta = \left(\frac{2\pi}{T}\right)^2\frac{\pi}{36}\sin\omega t$. Use this information to sketch the amplitude and phase responses (i.e. vs ω) of the steady-state oscillation. Use the formulae for damped SHM on page 1 without proof.

(ix) If the children's swing in (vii) was critically damped, determine the equation of motion $\theta(t)$, assuming that the swing is released from rest at $\theta = \frac{\pi}{6}$ radians. Assume the sinusoidal driving force machine has been turned off.

(x) Show that the transient responses $x(t)$ for (a) underdamped; (b) overdamped; (c) critically damped SHM systems described on page 1 are indeed solutions to $\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0$. *Tedious calculus but good for you!*

Question2 A pendulum of length l consists of a light inextensible string, supporting a small mass m. The top of the string is fixed in position, and the only significant forces on the mass are gravity (with strength g) and the tension in the string.

(i) Draw a diagram to represent the situation and then write down Newton II in radial and tangential directions. Show that for small angles, where $\sin\theta \approx \theta$, the motion is SHM with period $\tau = 2\pi\sqrt{l/g}$.

(ii) What will a graph of $\dot{\theta}$ vs $\omega\theta$ look like for the (SHM) motion? What would happen if there was damping? Assume the pendulum is released from rest i.e. $\theta(t) = \theta_0\cos\omega t$.

(iii) *Without* using the small angle approximation, use energy conservation to derive an expression of how the speed of the mass varies with θ, and sketch this. Determine an equation for the θ variation of string tension T, and sketch this too. Assume there is no damping and therefore no loss of energy.

Question 3 A charge of mass m is placed in the centre of a vacuum tube of length l. At either ends of the tube are identically charged plates, of the same sign as the charge on the mass. The force on the charge due to the left plate is k/x^2 where x is the distance from the left plate to the charge i.e. an inverse-square law.

(i) Explain why: $m\ddot{x} = kx^{-2} - k(l-x)^{-2}$

(ii) If $x = \frac{1}{2}l + z$, where z is a small ($|z| \ll l$) perturbation from equilibrium, show (using appropriate binomial expansions) that $\ddot{z} \approx -\dfrac{32k}{ml^3}z$.

Hence determine an expression for the frequency of oscillations of the charged mass m.

(iii) Evaluate the oscillation frequency (and equivalent EM wavelength in nm) using the following atomic-scale values.

$l = 1.23\times10^{-10}\,\text{m}$, $m = 1.67\times10^{-27}\,\text{kg}$, $k = \dfrac{e^2}{4\pi\varepsilon_0}$ where $e = 1.602\times10^{-19}\,\text{C}$, $\varepsilon_0 = 8.85\times10^{-12}\,\text{Nm}^2\text{C}^{-2}$.

Question 4 A cylinder of radius r and height h It is placed in a liquid of density ρ, whereupon it partially submerges to depth $x = \alpha h$ where $0 < \alpha < 1$. What is the mass m of the cylinder in terms of ρ, r, h ?

If the cylinder is displaced by extra depth $z(t=0) = \beta h$, where $\beta \ll \alpha$, determine an expression for the period T of the ensuing oscillations, i.e. the time variation of 'extra depth' z. Assume a uniform strength of gravity g.

Determine the period T of these oscillations (in s) for a cylindrical iceberg of radius 100m and height 200m. The density of seawater is 1025kg/m^3 and the density of ice is 920kg/m^3. $g = 9.81\text{m/s}^2$.

Question 5 *de-Moivre's theorem* states that $e^{i\theta} = \cos\theta + i\sin\theta$, which means $A\cos(\omega t - \phi)$ is the *real part* of $z = Ae^{i(\omega t-\phi)}$. Consider a 'complex' version of the driven SHM equation: $\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = A_0\omega_0^2 e^{i\omega t}$. By assuming a steady state solution $z(t) = Ae^{i(\omega t-\phi)}$, substitute into the SHM equation and hence prove:

$$A = A_0\omega_0^2 \Big/ \sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2} \qquad \text{and} \qquad \phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2 - \omega^2}\right).$$

Also prove that the maximum oscillation amplitude $A_{max} = \dfrac{A_0\omega_0^2}{2\gamma\sqrt{\omega_0^2 - \gamma^2}}$ when $\omega = \sqrt{\omega_0^2 - 2\gamma^2}$.

Question 6 (Adapted from a question in OCR *Mechanics 4* textbook in the *Small Oscillations & Stability* chapter).

A rigid rod of mass m and length $2a$ is suspended from a frictionless hinge. When the rod is hung in equilibrium, the bottom of the rod is a from the ground. At a point on the ground directly below the hinge is a small fixed ring. A light elastic cord of unstretched length $2a$ and spring constant $k = 5mg/2a$ is threaded though the rod and securely attached to the ring. The rod is displaced by a small angle, and oscillations of period T ensue.

(i) Show that the moment of inertia of the rod rotating about the hinge is $I = \frac{4}{3}ma^2$.

(ii) Show that the extension of the elastic is $x = a\sqrt{13 - 12\cos\theta}$, where θ is the angle of the rod from the vertical.

(iii) Show that the total energy of the system is: $E = \frac{2}{3}ma^2\dot{\theta}^2 + mga\left(17\tfrac{1}{4} - 16\cos\theta\right)$.

(iv) If no energy is lost (i.e. $\dot{E} = 0$) show that $T \approx 2\pi\sqrt{\dfrac{a}{12g}}$ for small oscillations.

(v) Show that the period of the rod swinging *without* the elastic is exactly four times that with the elastic.