STEP I 2000

1/ $\log_{10} 2 = 0.301029996$

$\log_{10} 3 = 0.477121255$

(i)

Note $\log_{10} 5 = \log_{10}\left(\frac{10}{2}\right)$

$= 1 - \log_{10} 2$

$= 0.698970004$

$= \boxed{0.699 \text{ to } 3.\text{dp}}$

$\log_{10} 6 = \log_{10}(3 \times 2)$

$= \log_{10} 2 + \log_{10} 3$

$= 0.778151251$

$= \boxed{0.778 \text{ to } 3.\text{dp}}$

So you could do this without a calculator

[What are the first k significant digits of b^n ?

$y = b^n$

$\log_{10} y = n \log_{10} b$ $\therefore \boxed{y = 10^{n \log_{10} b}}$

let $z = n\log_{10} b - \text{floor}(n \log_{10} b)$

i.e. the 'fractional part'.

First k digits are the first k digits of 10^{z+k-1}

This works since:

$y = 10^z \times \underbrace{10^{\text{floor}(n\log_{10} b)}}_{}$

This just adds extra zeros!]

Now consider:

$5 \times 10^{47} = 10^{\log_{10}(5 \times 10^{47})}$ [i.e. $x = 10^{\log_{10} x}$]

$= 10^{\log_{10} 5 + 47}$

$\simeq \boxed{10^{47.699}}$

$3^{100} = 10^{\log_{10} 3^{100}}$

$= 10^{100 \log_{10} 3}$

$\approx \boxed{10^{47.712}}$

Hence $\boxed{5 \times 10^{47} < 3^{100} < 6 \times 10^{47}}$

$6 \times 10^{47} = 10^{\log_{10}(6 \times 10^{47})}$

$= 10^{\log_{10} 6 + 47}$

$\approx \boxed{10^{47.778}}$

Q1(i)

(ii) let $y = 2^{1000}$

$y = 10^{\log_{10} 2^{1000}}$

$y = 10^{1000 \log_{10} 2}$

$1000 \log_{10} 2 = 301.029996\ldots$

So $y = 10^{0.02996\ldots} \times 10^{301}$

$\boxed{y = 1.0715086\ldots \times 10^{301}}$

So first digit is $\boxed{1}$

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let $y = 2^{10,000}$

$y = 10^{\log_{10} 2^{10,000}}$

$y = 10^{10,000 \log_{10} 2}$

$10,000 \log_{10} 2 = 3010.29996\ldots$

So $y = 10^{0.29996\ldots} \times 10^{3010}$

$\boxed{y = 1.99506312\ldots \times 10^{3010}}$

So first digit is $\boxed{1}$

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let $y = 2^{100,000}$, $y = 10^{\log_{10} 2^{100,000}}$, $y = 10^{100,000 \log_{10} 2}$

$100,000 \log_{10} 2 = 30,102.9996\ldots$

So $y = 10^{0.9996\ldots} \times 10^{30,102}$

$\boxed{y = 9.9900209\ldots \times 10^{30,102}}$ So first digit is $\boxed{9}$

Q1 (2) [See my MATLAB function $y = SF(b, x)$ for b^x in standard form]

2/ Consider $y = \left(x^4 - \frac{1}{x^2}\right)^5 \left(x - \frac{1}{x}\right)^6$

This will be a polynomial $y = x^{26} + \dots + x^{-16}$

To find the coefficient of x^{-12} we need to find n, m

s.t.
$$\left| (x^4)^n \left(-\frac{1}{x^2}\right)^{5-n} (x)^m \left(-\frac{1}{x}\right)^{6-m} \right| = x^{-12}$$

$\therefore$ $4n + 2n - 10 + m + m - 6 = -12$

$6n + 2m = 4$

$\boxed{3n + m = 2}$ Also $0 \le n \le 5$, $0 \le m \le 6$

Only solution is $n = 0$, $m = 2$

$\therefore$ x^{-12} term is $\binom{5}{0} (x^4)^0 \left(-\frac{1}{x^2}\right)^5 \times \binom{6}{2} (x)^2 \left(-\frac{1}{x}\right)^4$

$= -x^{-10} \times 15 x^2 x^{-4} = \boxed{-15x^{-12}}$

$\left[\binom{6}{2} = \frac{6!}{4!2!} = \frac{6 \times 5 \times 4!}{4!2!} = 3 \times 5 = 15 \right]$

Similarly, to find the coefficient of x^2 we need to find n, m

s.t.
$$\left| (x^4)^n \left(-\frac{1}{x^2}\right)^{5-n} (x)^m \left(-\frac{1}{x}\right)^{6-m} \right| = x^2$$

$0 \le n \le 5$
$0 \le m \le 6$

$\therefore$ $4n + 2n - 10 + m + m - 6 = 2$

$6n + 2m = 18$

$\boxed{3n + m = 9}$

Solutions are: $m = 0$, $n = 3$
$m = 6$, $n = 1$
$m = 3$, $n = 2$

Now coefficient of term $x^{6n + 2m - 16}$ is

$(-1)^{5 - n + 6 - m} \binom{5}{n} \binom{6}{m}$

Q2(i)

$\therefore$ coefficient of x^2 is

$$\left[\begin{matrix} n=1, & m=6 \\ n=2, & m=3 \\ n=3, & m=0 \end{matrix}\right] \qquad (-1)^{\overset{11-1-6=4}{4}}\binom{5}{1}\binom{6}{6} + (-1)^{\overset{11-2-3=6}{6}}\binom{5}{2}\binom{6}{3} + (-1)^{\overset{11-3-0=8}{8}}\binom{5}{3}\binom{6}{0}$$

$$= 5 + 200 + 10$$

$$= \boxed{215}$$

Now consider $y = (x^2-1)^{11}(x^4+x^2+1)^5$

[Not exactly an 'extension' of the first part!!]

→ Aim for a product of binomial expansions as before

$\therefore$ $y = (x^2-1)^{11}(x^2(x^2+1)+1)^5$

Goal is to find the coefficients of $\boxed{x^4}$ and $\boxed{x^{38}}$

$$y = (x^2-1)^{11}\{x^{10}(x^2+1)^5 + 5x^8(x^2+1)^4 + 10x^6(x^2+1)^3 + \dots$$
$$\dots + 10x^4(x^2+1)^2 + 5x^2(x^2+1) + 1\}$$

Now to get x^4, the $(x^2-1)^{11}$ is irrelevant apart from $(-1)^{11}$

So the coefficient is $-1(10+5) = \boxed{-15}$

To get x^{38}: options are

$$\binom{11}{0}x^{22} \times ?\,x^{16}$$
$$-\binom{11}{1}x^{20} \times ?\,x^{18}$$
$$\binom{11}{2}x^{18} \times ?\,x^{20}$$
$$-\binom{11}{3}x^{16} \times ?\,x^{22}$$

Since $(x^2-1)^{11}$
$$= \binom{11}{0}x^{22} + \binom{11}{1}(x^2)^{10}(-1)^1 + \dots$$

Now in $\{\dots\}$ highest power of x is x^{20} so in $\{\dots\}$

coefficients of
x^{20} are 1
x^{18} are $\binom{5}{4} = 5$
x^{16} are $\binom{5}{3} + 5 \times \binom{4}{0} = 15$

Q2 (2)

$\therefore$ overall coefficient of x^{38} is

$$(55)(1) - 11(5) + (1)(15) = \boxed{15}$$

$$\binom{11}{2}x^{18} \times \underset{1}{?}\,x^{20} \qquad -\binom{11}{1}x^{20} + \underset{\binom{5}{4}}{?}\,x^{18} \qquad \binom{11}{0}x^{22} + \underset{15}{?}\,x^{16}$$

$$15 = \binom{5}{3} + 5\binom{4}{0}$$

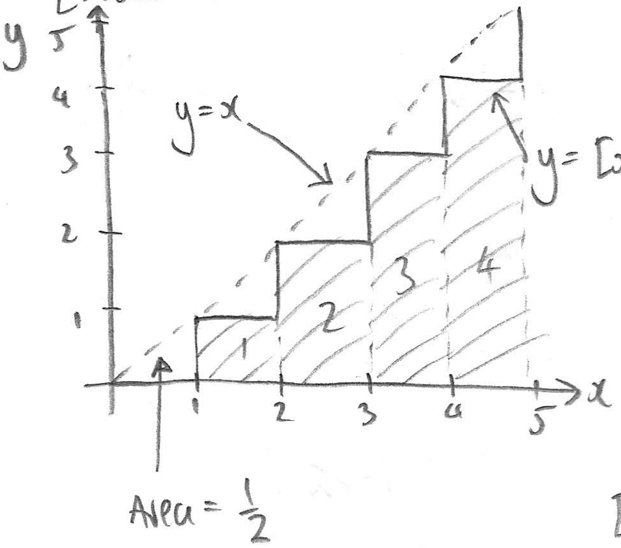
[From Wolfram Alpha:

$$(x^2-1)^{11}(x^4+x^2+1)^5 = x^{42} - 6x^{40} + 15x^{38} - 25x^{36} + 45x^{34}$$
$$- 81x^{32} + 111x^{30} - 135x^{28} + 180x^{26}$$
$$- 215x^{24} + 210x^{22} - 210x^{20} + 215x^{18}$$
$$- 180x^{16} + 135x^{14} - 111x^{12} + 81x^{10}$$
$$- 45x^{8} + 25x^{6} - 15x^{4} + 6x^{2} - 1$$

]

3/ let $[x]$ mean "round down x to the nearest integer".

[Note this is the same as the MATLAB floor(x) function.]



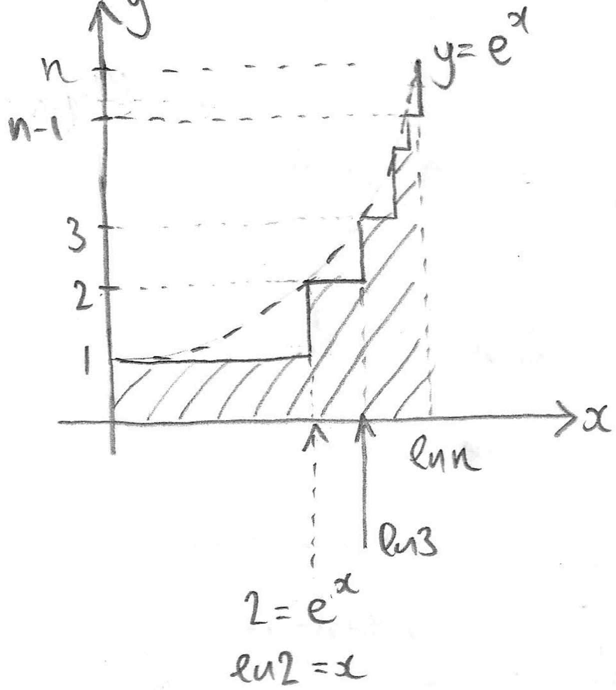
$$\therefore \int_0^5 [x]\,dx$$

$$= \tfrac{1}{2}(5)(5) - 5\left(\tfrac{1}{2}\right)$$

$$= \tfrac{5}{2}(5-1)$$

$$= \boxed{10}$$

[or simply $1+2+3+4 = 10$]



when $x = \ln n$, n integer

$e^x = e^{\ln n} = n$.

Hence from graph:

$$I = \int_0^{\ln n} [e^x]\,dx$$

$$= 1\times\ln 2 + 2\times(\ln 3 - \ln 2)$$
$$+ 3\times(\ln 4 - \ln 3)$$
$$+ 4\times(\ln 5 - \ln 4)$$
$$+ \ldots$$
$$+ (n-2)(\ln(n-1) - \ln(n-2))$$
$$+ (n-1)(\ln n - \ln(n-1))$$

$$= -\ln 2 - \ln 3 - \ln 4 - \ln 5 - \ln(n-1) + (n-1)\ln n$$

$$= -(\ln 2 + \ln 3 + \ln 4 + \ln 5 + \ldots + \ln n) + n\ln n$$

$$= -\ln(2\times 3\times 4\times 5\times \ldots n) + n\ln n$$

$$\Rightarrow \boxed{I = n\ln n - \ln n!}$$

Q3 (i)

So $\int_0^{\ln n} [e^x]\,dx = n\ln n - \ln n!$ as required

Now if $n \to \infty$ $\int_0^{\ln n} [e^x]\,dx \to \int_0^{\ln n} e^x\,dx$

ie steps get closer and closer together

(Step width is $\ln n - \ln(n-1) = \ln\left(\frac{n}{n-1}\right)$

$= \ln\left(\frac{1}{1-\frac{1}{n}}\right)$

So as $n \to \infty$, $\ln\left(\frac{1}{1-\frac{1}{n}}\right) \to \ln\left(\frac{1}{1}\right)$ which is zero)

$\therefore$ For large n $\left[e^x\right]_0^{\ln n} \approx n\ln n - \ln n!$

$\Rightarrow e^{\ln n} - 1 \approx n\ln n - \ln n!$

$\Rightarrow n - 1 \approx n\ln n - \ln n!$

Since $n >> 1$ $\Rightarrow \ln n! \approx n\ln n - n$

$\Rightarrow n! \approx e^{n\ln n - n}$

$\Rightarrow n! \approx e^{-n}\left(e^{\ln n}\right)^n$

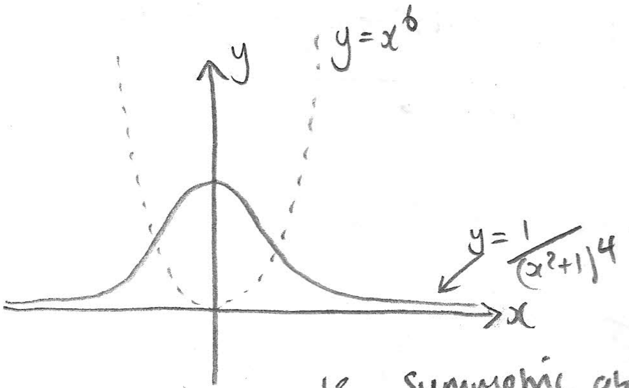
$\Rightarrow \boxed{n! \approx e^{-n} n^n}$ [$n>>1$]

This is $\boxed{\text{Stirling's Approximation}}$

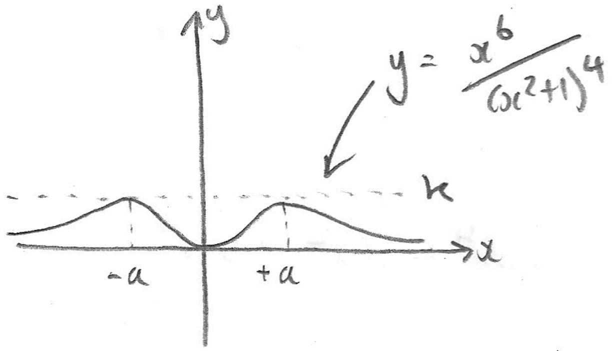
{ A more precise version is $n! \approx \sqrt{2\pi n}\, e^{-n} n^n$ }

Q3 (2)

4/ (i) $y = \dfrac{x^6}{(x^2+1)^4}$



ie symmetric about y axis



Expect maxima at $\pm a$ of $y=k$.

"largest value" is y s.t. $\dfrac{dy}{dx} = 0$, and this point being a maximum of y vs x

$$\frac{dy}{dx} = \frac{(x^2+1)^4(6x^5) - x^6(4(x^2+1)^3(2x))}{(x^2+1)^8}$$

$$\frac{dy}{dx} = \frac{6x^5(x^2+1)^4 - 8x^7(x^2+1)^3}{(x^2+1)^8}$$

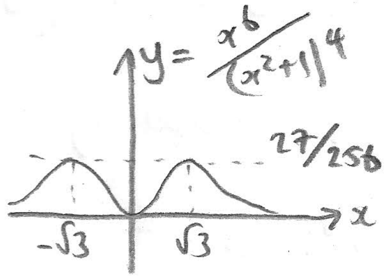
$$\frac{dy}{dx} = \frac{6x^5(x^2+1) - 8x^7}{(x^2+1)^5}$$

$$\frac{dy}{dx} = \frac{6x^7 + 6x^5 - 8x^7}{(x^2+1)^5}$$

$$\boxed{\frac{dy}{dx} = \frac{6x^5 - 2x^7}{(x^2+1)^5}}$$

$\therefore \dfrac{dy}{dx} = 0$ when

$$6x^5 - 2x^7 = 0$$
$$6x^5 = 2x^7$$
$$3 = x^2$$
$$\therefore \boxed{x = \pm\sqrt{3}}$$



$x^6 = (x^2)^3 = 3^3 = 27$

$x^2 = 3$

$\therefore k = \dfrac{27}{4^4}$

$k = 27/256$

≈ 0.105

So, in range $0 \le x \le 1$ y is increasing.

$\therefore$ largest value is when $x = 1$

ie $y = \dfrac{1}{(1+1)^4} = \boxed{\dfrac{1}{16}}$

Q4 (i)

(ii) let $\dfrac{1}{(x^2+1)^4} = \dfrac{d}{dx}\left(\dfrac{Ax^5+Bx^3+Cx}{(x^2+1)^3}\right) + \dfrac{Dx^6}{(x^2+1)^4}$ (†)

$$\frac{d}{dx}\left(\frac{Ax^5+Bx^3+Cx}{(x^2+1)^3}\right) = \frac{(x^2+1)^3(5Ax^4+3Bx^2+C) - 3(x^2+1)^2(2x)\,(*)}{(x^2+1)^6}$$

$$(*) = Ax^5+Bx^3+Cx$$

$$= \frac{(x^2+1)(5Ax^4+3Bx^2+C) - 6x(Ax^5+Bx^3+Cx)}{(x^2+1)^4}$$ ← ie ÷ numerator and denominator by x^2+1

So for (†) to be true

$$1 = (x^2+1)(5Ax^4+3Bx^2+C) - 6Ax^6 - 6Bx^4 - 6Cx^2 + Dx^6$$

$$1 = 5Ax^6 + 5Ax^4 + 3Bx^4 + 3Bx^2 + Cx^2 + C$$
$$- 6Ax^6 - 6Bx^4 - 6Cx^2 + Dx^6$$

$\therefore\ 0 = x^6(D-A) + x^4(5A-3B) + x^2(3B-5C) + C - 1$

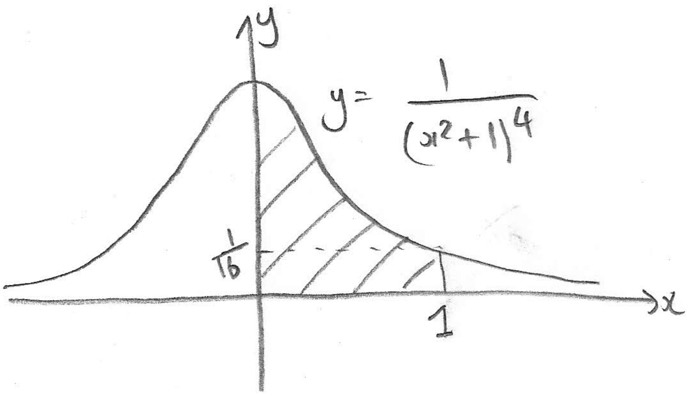
$\therefore$ comparing coefficients (since above must be true $\forall x$)

x^6: $D = A$
x^4: $5A = 3B$
x^2: $3B = 5C$
x^0: $C = 1$ $\quad\therefore\quad \boxed{B = 5/3} \quad \boxed{A = D = 1}$

So $$\boxed{\frac{1}{(x^2+1)^4} \equiv \frac{d}{dx}\left(\frac{x^5+\frac{5}{3}x^3+x}{(x^2+1)^3}\right) + \frac{x^6}{(x^2+1)^4}}$$

Q4 2

(iii) Now consider $I = \int_0^1 \frac{1}{(x^2+1)^4}\,dx$



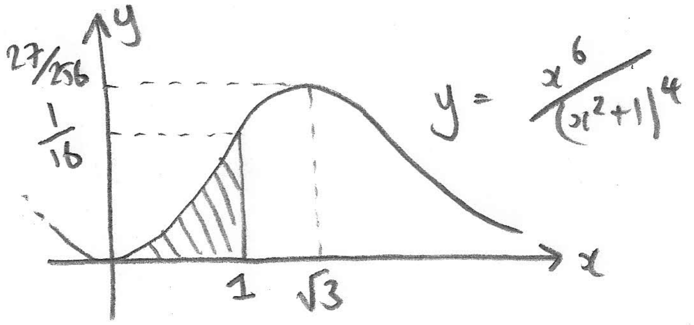
$$\therefore \int_0^1 \frac{1}{(x^2+1)^4}\,dx = \int_0^1 \frac{d}{dx}\left(\frac{x^5+\frac{8}{3}x^3+x}{(x^2+1)^3}\right)dx + \int_0^1 \frac{x^6}{(x^2+1)^4}\,dx$$

$$= \left[\frac{x^5+\frac{8}{3}x^3+x}{(x^2+1)^3}\right]_0^1 + \int_0^1 \frac{x^6}{(x^2+1)^4}\,dx$$

$$= \frac{1+\frac{8}{3}+1}{8} + \int_0^1 \frac{x^6}{(x^2+1)^4}\,dx$$

$$= \frac{11}{24} + \int_0^1 \frac{x^6}{(x^2+1)^4}\,dx$$

Now from part (i)



So $\int_0^1 \frac{x^6}{(x^2+1)^4}\,dx < \frac{1}{16}$

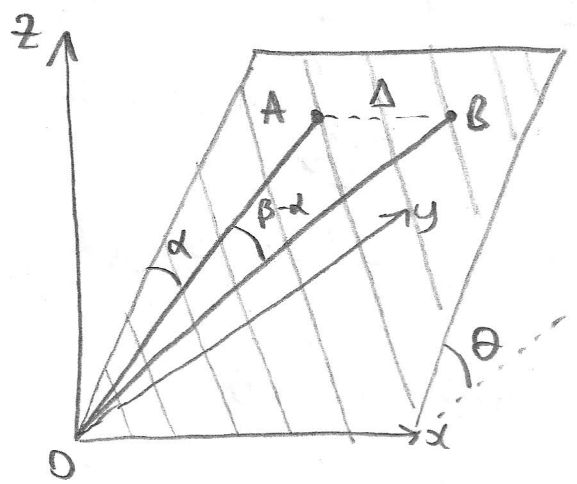
$$\boxed{\frac{11}{24} < \int_0^1 \frac{1}{(x^2+1)^4}\,dx < \frac{11}{24} + \frac{1}{16}}$$

{ Surely $\leq$ is wrong in paper. Must be < }

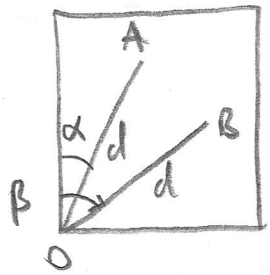
$$\left[\, 0.458\dot{3} < \int_0^1 \frac{1}{(x^2+1)^4}\,dx < 0.5208\dot{3} \,\right]$$

Q4③

5/



A and B walk on an inclined plane a distance d

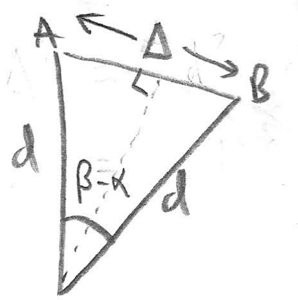


(looking down on plane)

in x, y, z coordinates A is $d\begin{pmatrix}\sin\alpha \\ \cos\alpha\cos\theta \\ \cos\alpha\sin\theta\end{pmatrix}$

B is $d\begin{pmatrix}\sin\beta \\ \cos\beta\cos\theta \\ \cos\beta\sin\theta\end{pmatrix}$

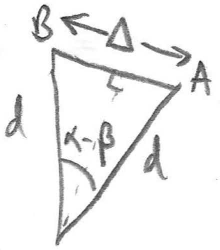
Distance between A and B is Δ.



$\therefore \frac{\Delta}{2} = d\sin\left(\frac{1}{2}(\beta-\alpha)\right)$

$\therefore \Delta = 2d\sin\left(\frac{1}{2}(\beta-\alpha)\right)$ $(\beta > \alpha)$

If $\alpha > \beta$



so $\frac{\Delta}{2} = d\sin\left(\frac{1}{2}(\alpha-\beta)\right)$

Since $\sin(-x) = -\sin x$ we can write generally

$$\boxed{\Delta = 2d\left|\sin\tfrac{1}{2}(\alpha-\beta)\right|}$$

as required.

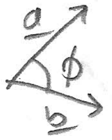
Now line joining A to B has vector

$$\underline{\Delta} = d\begin{pmatrix}\sin\beta - \sin\alpha \\ (\cos\beta - \cos\alpha)\cos\theta \\ (\cos\beta - \cos\alpha)\sin\theta\end{pmatrix}$$

Q5 ①

let angle ϕ be between $\underline{\Delta}$ and $\hat{\underline{z}}$ (vertical)

$\therefore \quad |\underline{\Delta}|\,|\hat{\underline{z}}| \cos\phi = \underline{\Delta} \cdot \hat{\underline{z}}$



$|\underline{a}||\underline{b}|\cos\phi = \underline{a} \cdot \underline{b}$

$|\hat{\underline{z}}| = 1$ and from above $|\underline{\Delta}| = 2d\left|\sin\tfrac{1}{2}(\alpha-\beta)\right|$

$$\therefore \quad 2d\left|\sin\tfrac{1}{2}(\alpha-\beta)\right| \cos\phi = d(\cos\beta - \cos\alpha)\sin\theta$$

$$\therefore \quad \cos\phi = \sin\theta \frac{(\cos\beta - \cos\alpha)}{2\left|\sin\tfrac{1}{2}(\alpha-\beta)\right|}$$

Now, motivated by the answer $\left(\cos\phi = \sin\phi \sin\tfrac{1}{2}(\alpha+\beta) \right)$
$\alpha \neq \beta$

Consider: $\sin\tfrac{1}{2}(\alpha-\beta)\sin\tfrac{1}{2}(\alpha+\beta) = \sin\left(\tfrac{\alpha}{2} - \tfrac{\beta}{2}\right)\sin\left(\tfrac{\alpha}{2} + \tfrac{\beta}{2}\right)$

$$= \left(\sin\tfrac{\alpha}{2}\cos\tfrac{\beta}{2} - \cos\tfrac{\alpha}{2}\sin\tfrac{\beta}{2}\right)\left(\sin\tfrac{\alpha}{2}\cos\tfrac{\beta}{2} + \cos\tfrac{\alpha}{2}\sin\tfrac{\beta}{2}\right)$$

$$= \sin^2\tfrac{\alpha}{2}\cos^2\tfrac{\beta}{2} - \cos^2\tfrac{\alpha}{2}\sin^2\tfrac{\beta}{2} \qquad = *$$

Now $\sin^2 A = \tfrac{1}{2}(1 - \cos 2A)$
$\cos^2 A = \tfrac{1}{2}(1 + \cos 2A)$

$\therefore \quad \sin^2\tfrac{\alpha}{2} = \tfrac{1}{2}(1 - \cos\alpha)$
$\cos^2\tfrac{\alpha}{2} = \tfrac{1}{2}(1 + \cos\alpha)$

$$\therefore \quad * = \tfrac{1}{4}(1 - \cos\alpha)(1 + \cos\beta) - \tfrac{1}{4}(1 + \cos\alpha)(1 - \cos\beta)$$

$$= \tfrac{1}{4}\left\{ \cancel{1} - \cos\alpha + \cos\beta - \cancel{\cos\alpha\cos\beta} - \cancel{1} - \cos\alpha + \cos\beta + \cancel{\cos\alpha\cos\beta} \right\}$$

$$= \tfrac{1}{2}(\cos\beta - \cos\alpha)$$

as (2)

So $\quad \cos\beta - \cos\alpha = 2\sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\alpha+\beta)$

$$\therefore \quad \cos\phi = \sin\theta \underbrace{\frac{\sin\frac{1}{2}(\alpha-\beta)}{\left|\sin\frac{1}{2}(\alpha-\beta)\right|}}_{= \text{Sign}(\alpha-\beta)} \sin\frac{1}{2}(\alpha+\beta)$$

So if $\sin\frac{1}{2}(\alpha-\beta) > 0$ then

$$\boxed{\cos\phi = \sin\theta\sin\tfrac{1}{2}(\alpha+\beta)}$$ as required

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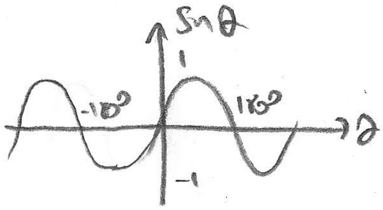
Now A and B always walk uphill (and right of 0) $\therefore \quad 0 < \alpha, \beta < 90°$

well 'same side of'

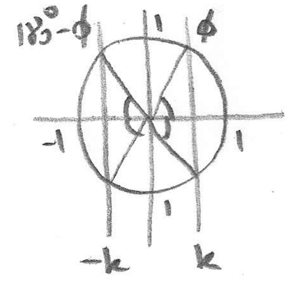
$\therefore \quad |\alpha-\beta| < 90°$

$\Rightarrow \quad -\frac{1}{\sqrt{2}} < \sin\frac{1}{2}(\alpha-\beta) < \frac{1}{\sqrt{2}} \qquad \left(\sin 45° = \frac{1}{\sqrt{2}}\right)$



This also explains why $\sin\frac{1}{2}(\alpha+\beta) > 0$

$\cos\phi = \pm k$



From "$\cos\phi$ is the x coordinate of the unit circle"
If we only accept the acute angle for ϕ (ie use $180° - \phi$ if $\phi > 90°$)

$\cos\phi = -k \Rightarrow \phi$ is chosen
$\cos\phi = k \Rightarrow \phi$ is chosen also

$\therefore$ since $\sin\theta\sin\frac{1}{2}(\alpha+\beta) > 0$

ϕ is returned from $\boxed{\cos\phi = \sin\theta\sin\frac{1}{2}(\alpha+\beta)}$ regardless of whether $\alpha > \beta$ or $\beta > \alpha$

Q5 3

6/ $x^2 - y^2 + x + 3y - 2 = (x-y+2)(x+y-1)$ [conjecture!]

$$(x-y+2)(x+y-1) = \begin{array}{c|c|c|c} & x & -y & 2 \\ \hline x & x^2 & -xy & 2x \\ \hline y & xy & -y^2 & 2y \\ \hline -1 & -x & y & -2 \end{array}$$

$$= \boxed{x^2 - y^2 + 3y + x - 2} \quad ✓$$

Now consider the region in the x,y plane satisfied by

$$x^2 - y^2 + x + 3y > 2$$

$$\Rightarrow \quad x^2 - y^2 + x + 3y - 2 > 0$$

$$\Rightarrow \quad (x-y+2)(x+y-1) > 0 \qquad \text{(from above)}$$

This means $x-y+2 > 0$ & $x+y-1 > 0$ (A)

or// $x-y+2 < 0$ & $x+y-1 < 0$ (B

(A) $x-y+2 > 0$

$\Rightarrow \boxed{x+2 > y}$

(B) $x-y+2 < 0$

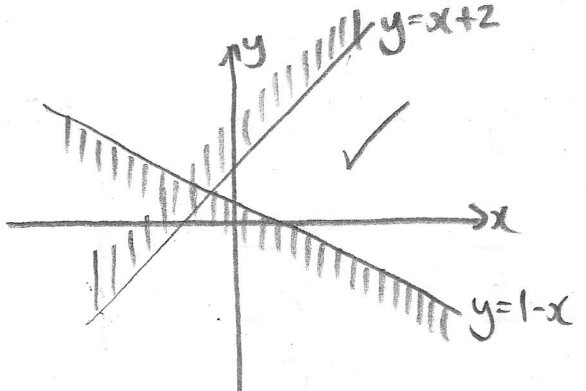
$\Rightarrow \boxed{x+2 < y}$

(A) $x+y-1 > 0$

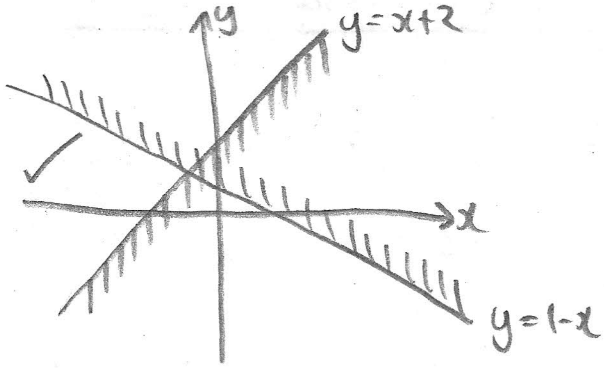
$\Rightarrow \boxed{y > 1-x}$

(B) $x+y-1 < 0$

$\Rightarrow \boxed{y < 1-x}$



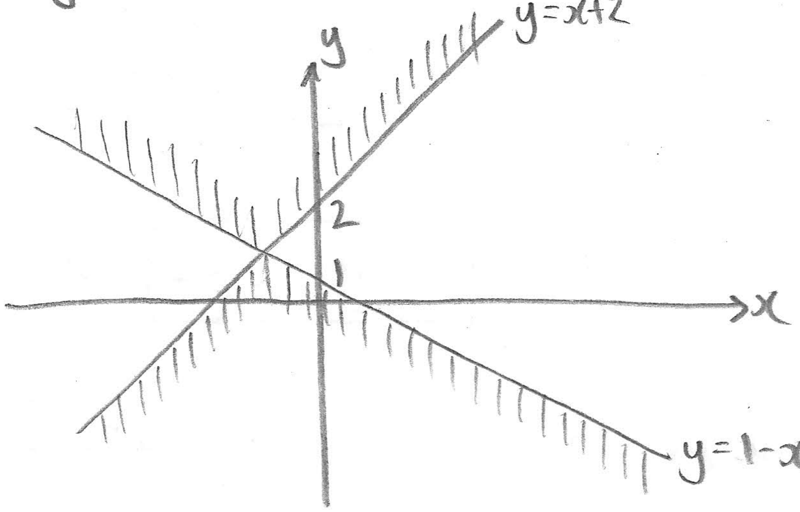
(A)



(B)

Q6 ①

$\therefore$ Valid region is unshaded below
(note 'or' condition means we can add the valid regions)



Now consider $x^2 - 4y^2 + 3x - 2y < -2$

ie $x^2 - 4y^2 + 3x - 2y + 2 < 0$

Motivated by the method above, let us try and factorize $x^2 - 4y^2 + 3x - 2y + 2$ into two ( ) involving only linear terms in x and y.

Try $x^2 - 4y^2 + 3x - 2y + 2 = (x + ky + 1)(x - ky + 2)$

$$= \begin{array}{c|c|c|c} & x & -ky & 2 \\ \hline x & x^2 & -kxy & 2x \\ \hline ky & kxy & -k^2y^2 & 2ky \\ \hline 1 & x & -ky & 2 \end{array} = x^2 - k^2y^2 + 3x + ky + 2$$

$\therefore$ y^2: $4 = k^2$
y: $-2 = k$
so $\boxed{k = -2}$

$\therefore$ $x^2 - 4y^2 + 3x - 2y + 2 = (x - 2y + 1)(x + 2y + 2)$

$\therefore \quad x^2 - 4y^2 + 3x - 2y + 2 < 0$

$\Rightarrow \quad (x - 2y + 1)(x + 2y + 2) < 0$

$\therefore \quad x - 2y + 1 > 0 \quad \& \quad x + 2y + 2 < 0 \qquad (A)$

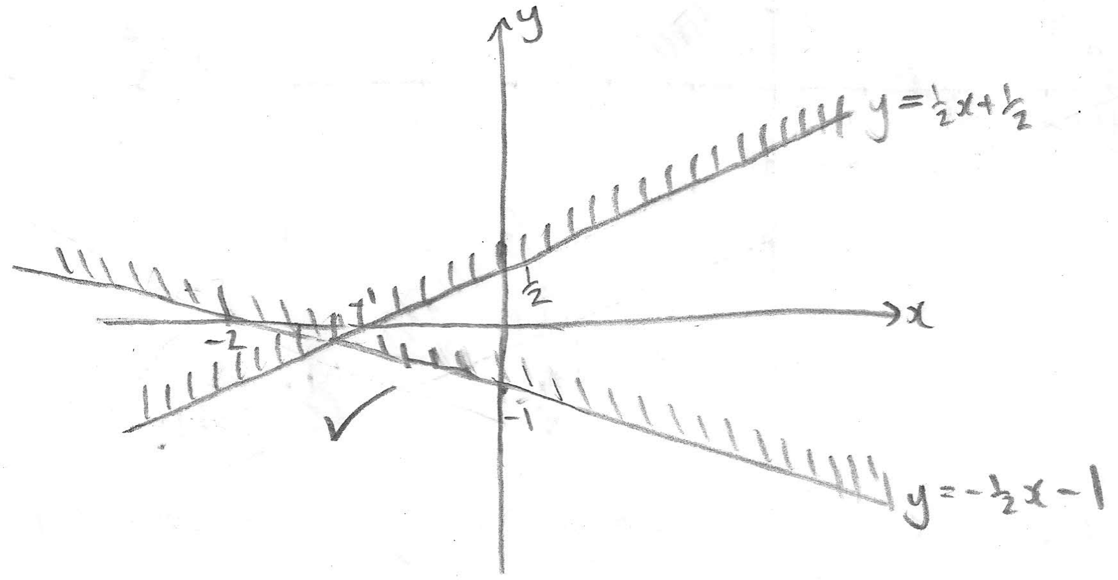
$\quad \frac{x+1}{2} > y \qquad\qquad y < -\frac{1}{2}x - 1$

or $\quad x - 2y + 1 < 0 \quad \& \quad x + 2y + 2 > 0 \qquad (B)$

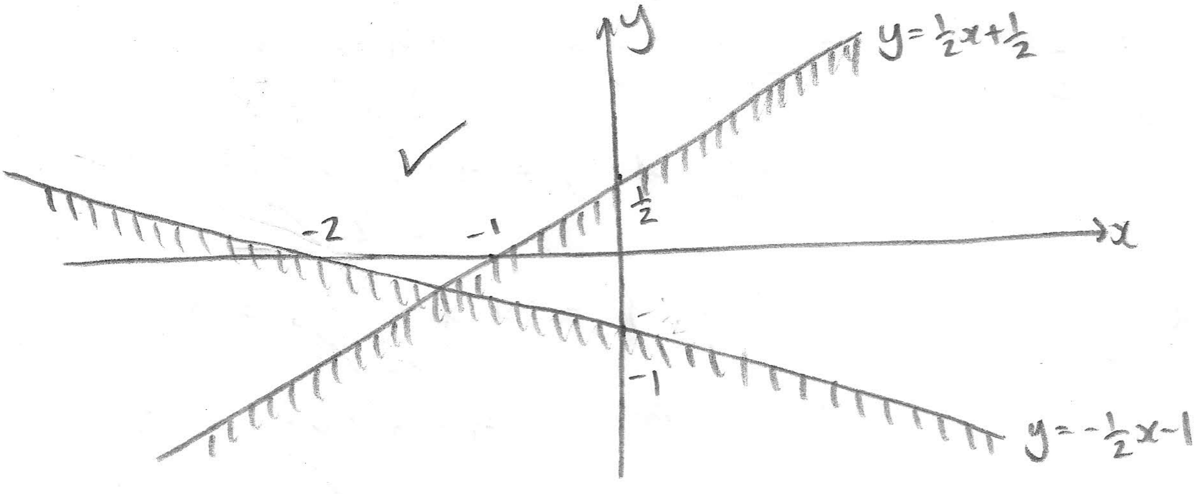
$\quad \frac{x+1}{2} < y \qquad\qquad y > -\frac{1}{2}x - 1$

Now $\quad x - 2y + 1 = 0 \quad \Rightarrow \quad y = \frac{1}{2}x + \frac{1}{2}$

$\quad x + 2y + 2 = 0 \quad \Rightarrow \quad y = -\frac{1}{2}x - 1$

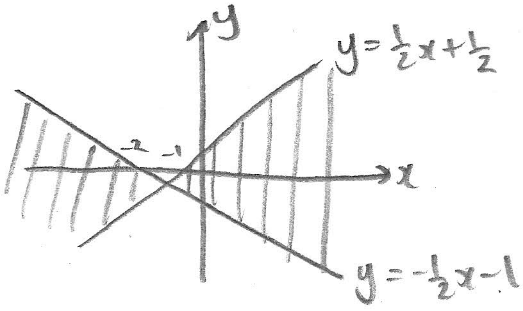


(A)



(B)

So overall region is unshaded



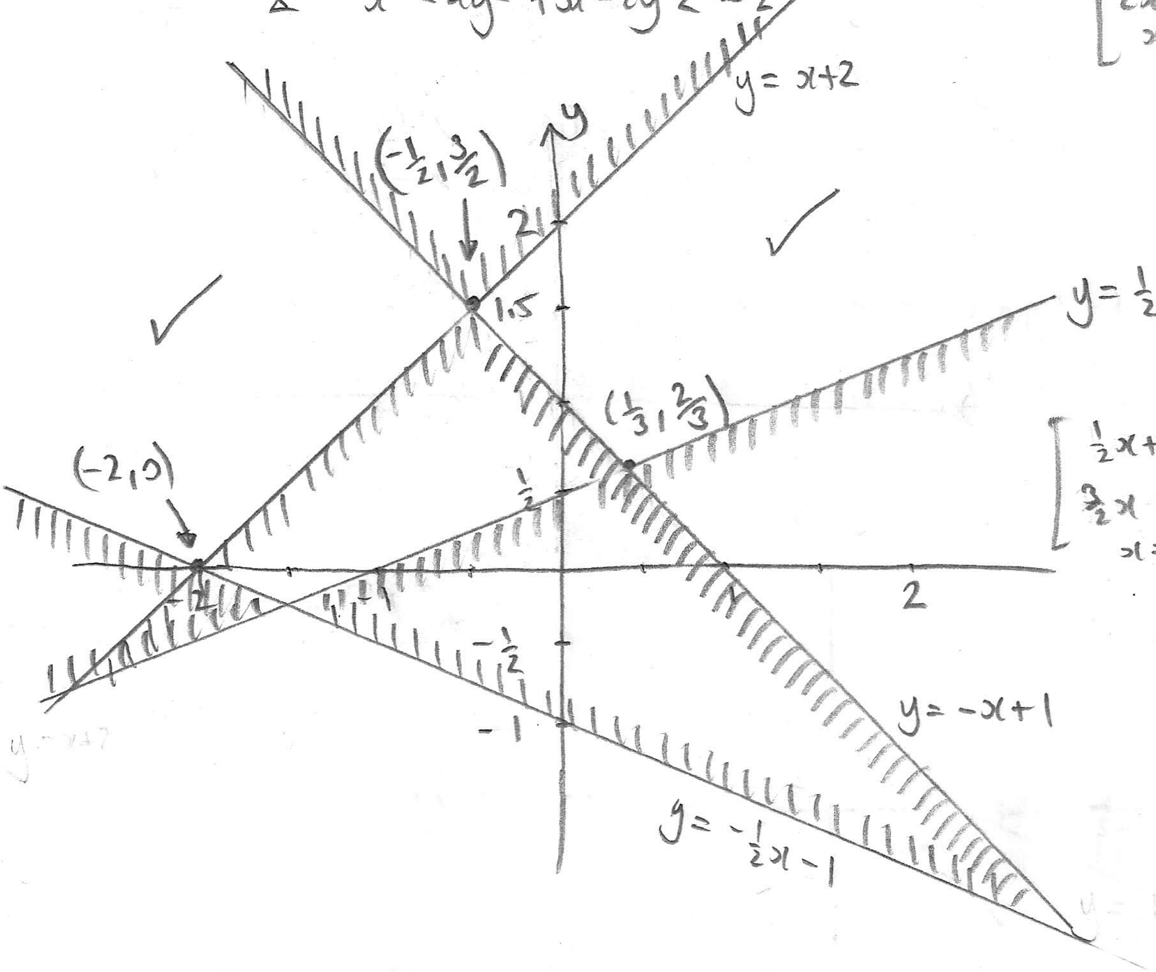
Q6 (3)

Now is there a point (or set of points) which satisfy

BOTH $x^2 - y^2 + x + 3y > 2$

& $x^2 - 4y^2 + 3x - 2y < -2$

$$\left[\begin{array}{l} x+2 = -x+1 \\ 2x = -1 \\ x = -\tfrac{1}{2} \end{array}\right]$$



$$\left[\begin{array}{l} \tfrac{1}{2}x + \tfrac{1}{2} = -x+1 \\ \tfrac{3}{2}x = \tfrac{1}{2} \\ x = \tfrac{1}{3} \end{array}\right]$$

So $(-2,2)$ or $(2,2)$ should satisfy both inequalities

check $x = -2$, $y = 2$ $\quad x^2 - y^2 + x + 3y = 4 - 4 - 2 + 6 = 4$
> 2 ✓
$x^2 - 4y^2 + 3x - 2y = 4 - 8 - 6 - 4 = -14$
< -2 ✓

check $x = 2$, $y = 2$ $\quad x^2 - y^2 + x + 3y = 4 - 4 + 2 + 6 = 8$
> 2 ✓
$x^2 - 4y^2 + 3x - 2y = 4 - 16 + 6 - 4 = -10$
< -2 ✓

Q6 (4)

7/ $f(x) = ax - \dfrac{x^3}{1+x^2}$

$$f'(x) = a - \left\{ \frac{(1+x^2)3x^2 - x^3(2x)}{(1+x^2)^2} \right\}$$

$$= a - \left\{ \frac{3x^2 + 3x^4 - 2x^4}{(1+x^2)^2} \right\}$$

$$= a - \frac{3x^2 + x^4}{(1+x^2)^2} \Rightarrow a = f'(x) + \frac{3x^2+x^4}{(1+x^2)^2}$$

Now if $a \geq 9/8$

$$\Rightarrow f'(x) + \frac{3x^2 + x^4}{(1+x^2)^2} \geq 9/8$$

$$f'(x) \geq \frac{9}{8} - \frac{3x^2+x^4}{(1+x^2)^2}$$

$$f'(x) \geq \frac{9(1+x^2)^2 - 8(3x^2+x^4)}{8(1+x^2)^4}$$

$$f'(x) \geq \frac{9 + 18x^2 + 9x^4 - 24x^2 - 8x^4}{8(1+x^2)^4}$$

$$f'(x) \geq \frac{9 - 6x^2 + x^4}{8(1+x^2)^4}$$

$$f'(x) \geq \frac{(x^2-3)^2}{8(1+x^2)^4}$$

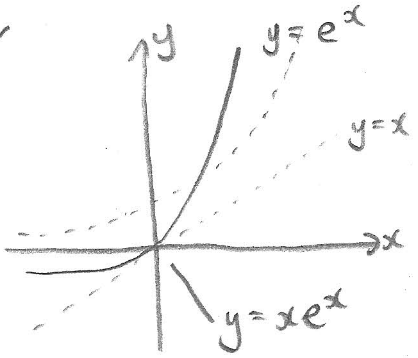
$$f'(x) \geq \frac{1}{8}\left(\frac{x^2-3}{(1+x^2)^2}\right)^2$$

So since $(\ldots)^2 \geq 0$ then $f'(x) \geq 0 \ \forall x$

forall.

Q7(i)

8/



$y = xe^x < 0$ for $x < 0$

$y = xe^x > 0$ for $x > 0$

$\therefore \int_{-1}^{1} |xe^x|\,dx = -\int_{-1}^{0} xe^x dx + \int_{0}^{1} xe^x\,dx$

$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + c = \boxed{e^x(x-1)+c}$

check: $\frac{d}{dx}(xe^x - e^x + c) = xe^x + e^x - e^x = xe^x$ ✓

$\left[\quad \int uv\,dx = \left(\int u\,dx\right) v - \int \left(\int u\,dx\right) \frac{dv}{dx}\,dx \quad \right]$

$\therefore \int_{-1}^{1} |xe^x|\,dx = -\left[e^x(x-1)\right]_{-1}^{0} + \left[e^x(x-1)\right]_{0}^{1}$

$= -\left[(-1) - (e^{-1}(-2))\right] + \left[(0) - (-1)\right]$

$= 1 - 2/e + 1$

$= \boxed{2 - 2/e}$

(i) $I = \int_0^4 |x^3 - 2x^2 - x + 2|\,dx$

$y = x^3 - 2x^2 - x + 2$ Need to factorize this to find where curve crosses the x axis and $\therefore$ -ve.

$x = 1$, $y = 1 - 2 - 1 + 2 = 0 \quad \therefore (x-1)$ is a factor

$\therefore x^3 - 2x^2 - x + 2 = (x-1)(x^2 + ax - 2)$

$= x^3 - x^2 + ax^2 - ax - 2x + 2$

$= x^3 + x^2(a-1) - x(2+a) + 2$

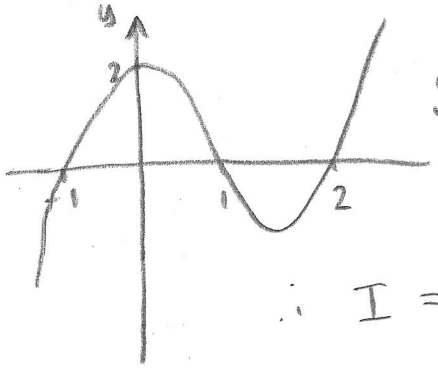
Comparing coefficients: $-2 = a - 1 \quad [x^2]$

$1 = 2 + a \quad [x]$ ie $\boxed{a = -1}$

Q8 ①

$y = (x-1)(x^2-x-2)$

$y = (x-1)(x+1)(x-2)$



$y = (x-1)(x+1)(x-2) = x^3 - 2x^2 - x + 2$

$\therefore \; I = \int_0^1 (x^3 - 2x^2 - x + 2)\,dx - \int_1^2 (x^3-2x^2-x+2)\,dx$

$+ \int_2^4 (x^3 - 2x^2 - x + 2)\,dx$

$= \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_0^1 - \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_1^2$

$+ \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_2^4$

$= \left[\frac{1}{4} - \frac{2}{3} - \frac{1}{2} + 2\right] + \left[\left(\frac{1}{4} - \frac{2}{3} - \frac{1}{2} + 2\right) - \left(\frac{16}{4} - \frac{2}{3}(8) - \frac{4}{2} + 4\right)\right]$

$+ \left[\left(\frac{256}{4} - \frac{2}{3}64 - \frac{16}{2} + 8\right) - \left(\frac{16}{4} - \frac{2}{3}(8) - \frac{4}{2} + 4\right)\right]$

$= 1\frac{1}{12} + 1\frac{1}{12} - \frac{2}{3} + 21\frac{1}{3} - 2/3$

$= \boxed{22\frac{1}{6}}$

Q8 2

(ii) $I = \int_{-\pi}^{\pi} |\sin x + \cos x| \, dx$

$\sin x + \cos x = R\sin(x+\alpha)$

$= R\sin x \cos\alpha + R\cos x \sin\alpha$

$\therefore$ Comparing coefficients

$\sin x$: $1 = R\cos\alpha$ ①

$\cos x$: $1 = R\sin\alpha$ ②

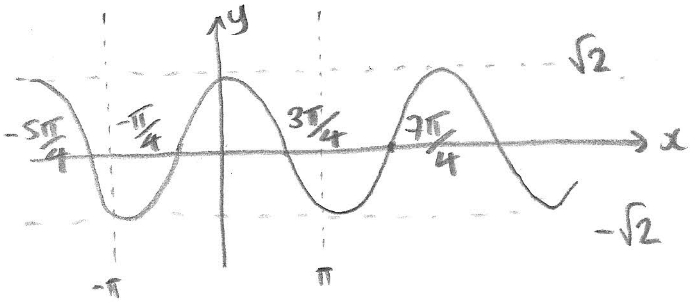
②/①: $\tan\alpha = 1 \Rightarrow \boxed{\alpha = \pi/4}$

①^2+②^2: $2 = R^2\cos^2\alpha + R^2\sin^2\alpha$

$\Rightarrow 2 = R^2(\cos^2\alpha + \sin^2\alpha)$

$\Rightarrow \pm\sqrt{2} = R$ Now must be true look since $\sin(0.1) + \cos(0.1) > 0$

$\therefore \boxed{\sin x + \cos x = \sqrt{2}\sin(x + \pi/4)}$



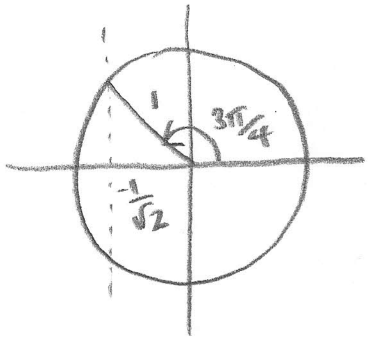
$\therefore I = -\int_{-\pi}^{-\pi/4} \sqrt{2}\sin(x+\frac{\pi}{4})\,dx + \int_{-\pi/4}^{3\pi/4} \sqrt{2}\sin(x+\frac{\pi}{4})\,dx$

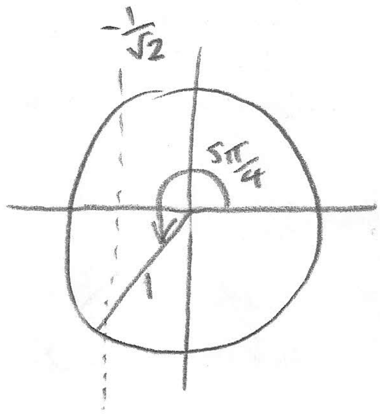
$- \int_{3\pi/4}^{\pi} \sqrt{2}\sin(x+\pi/4)\,dx$

$= -\sqrt{2}\left[-\cos(x+\frac{\pi}{4})\right]_{-\pi}^{-\pi/4} + \sqrt{2}\left[-\cos(x+\frac{\pi}{4})\right]_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} - \sqrt{2}\left[-\cos(x+\frac{\pi}{4})\right]_{\frac{3\pi}{4}}^{\pi}$

Q8 ③

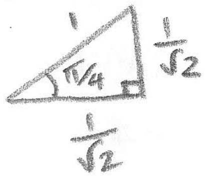
$$= -\sqrt{2}\left[-\cos(0) + \cos\left(-\frac{3\pi}{4}\right)\right] + \sqrt{2}\left[-\cos\pi + \cos(0)\right] - \sqrt{2}\left[-\cos\frac{5\pi}{4} + \cos\pi\right]$$





$\cos 0 = 1$

$\cos\pi = -1$



$\cos\left(-\frac{3\pi}{4}\right) = \cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$ $\quad$ $\cos\frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$

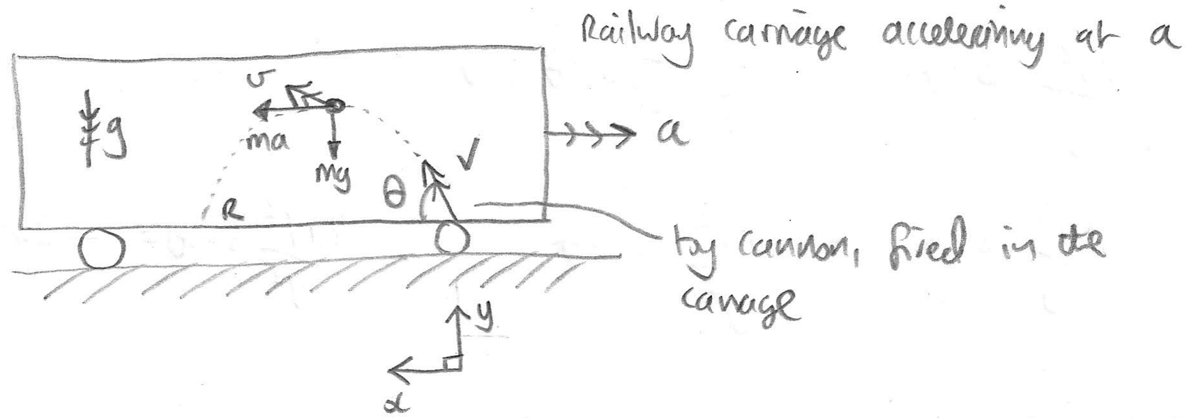
$$\therefore \quad I = -\sqrt{2}\left(-1 - \frac{1}{\sqrt{2}}\right) + \sqrt{2}\left(-(-1) + 1\right) - \sqrt{2}\left(\frac{1}{\sqrt{2}} - 1\right)$$

$$I = \sqrt{2} + 1 + \sqrt{2} + \sqrt{2} - 1 + \sqrt{2}$$

$$\therefore \quad I = \boxed{4\sqrt{2}}$$

$\boxed{Q8 \; 4}$

9/



Relative to the accelerating carriage, the cannon shell will 'feel' an additional force $\overset{\leftarrow}{ma}$ in addition to weight $\downarrow mg$

By Newton II, and ignoring air resistance

$$
\begin{aligned}
/\!/x: \quad m\ddot{x} &= ma \\
/\!/y: \quad m\ddot{y} &= -mg
\end{aligned}
\Bigg\} \text{ ie eqn of constant acceleration motion}
$$

$$\therefore \quad \dot{x} = V\cos\theta + at$$

$$x = Vt\cos\theta + \tfrac{1}{2}at^2$$

$$\dot{y} = V\sin\theta - gt$$

$$y = Vt\sin\theta - \tfrac{1}{2}gt^2$$

Maximum range R when $y = 0$,

$$y = t(V\sin\theta - \tfrac{1}{2}gt)$$

So this is clearly when

$$\tfrac{1}{2}gt = V\sin\theta$$

$$\Rightarrow \quad t = \frac{2V\sin\theta}{g}$$

$$\therefore \quad R = V\left(\frac{2V\sin\theta}{g}\right)\cos\theta + \frac{1}{2}a\left(\frac{2V\sin\theta}{g}\right)^2$$

$$\boxed{R = \frac{V^2}{g}\sin 2\theta + \frac{2a}{g^2}V^2\sin^2\theta}$$

$[2\sin\theta\cos\theta = \sin 2\theta]$

If V is a constant, find $\frac{dR}{d\theta} = 0$ to work out the maximum range if θ is variable

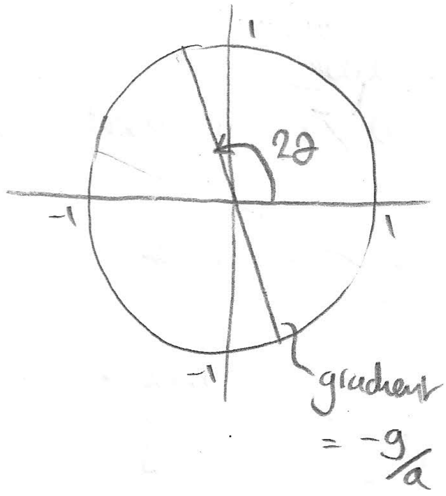
⊗ ①

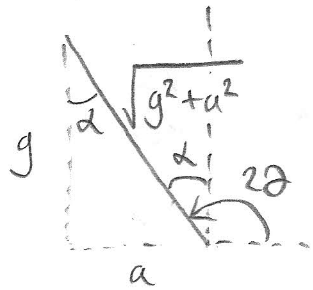
$$\frac{dR}{d\theta} = \frac{2v^2}{g}\cos 2\theta + \frac{4a}{g^2}v^2\sin\theta\cos\theta$$

$$= \frac{2v^2}{g}\cos 2\theta + \frac{2av^2}{g^2}\sin 2\theta \qquad \therefore \quad \frac{2av^2}{g^2}\sin 2\theta = -\frac{2v^2\cos 2\theta}{g}$$

Since $\tan 2\theta = \dfrac{\sin 2\theta}{\cos 2\theta}$

$$\Rightarrow \boxed{\tan 2\theta = -g/a}$$





$$2\theta = \frac{\pi}{2} + \alpha$$

if $g \gg a$ then $\alpha \ll 1$

$\tan\alpha = a/g$ $\quad \therefore$ when α is small $\tan\alpha \simeq \alpha$

So $2\theta \simeq \frac{\pi}{2} + a/g$

$$\therefore \boxed{\theta \simeq \frac{\pi}{4} + \frac{a}{2g}}$$

So, $R = \dfrac{v^2}{g}\sin 2\theta + \dfrac{2v^2}{g^2}a\sin^2\theta$ $\quad$ using $\theta \simeq \frac{\pi}{4} + \frac{a}{2g}$

and using $\dfrac{a}{2g} \ll 1$

$$\sin 2\theta \simeq \sin\left(\frac{\pi}{2} + \frac{a}{g}\right) = \sin\frac{\pi}{2}\cos\frac{a}{g} + \cos\frac{\pi}{2}\sin\frac{a}{g}$$

Now $\cos x \simeq 1 - x^2/2$ when $|x| \ll 1$ $\qquad \sin\frac{\pi}{2} = 1$, $\cos\frac{\pi}{2} = 0$

$\therefore \sin 2\theta \simeq 1 - a^2/g^2$

Q9 2

$$\sin\theta \simeq \sin\left(\frac{\pi}{4} + \frac{a}{2g}\right) = \sin\frac{\pi}{4}\cos\frac{a}{2g} + \cos\frac{\pi}{4}\sin\frac{a}{2g}.$$

$$\sin\tfrac{\pi}{4} = \cos\tfrac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \text{let } \sin\frac{a}{2g} \simeq \frac{a}{2g}$$

$$\therefore \quad \sin\theta \approx \frac{1}{\sqrt{2}}\left(1 - \frac{a^2}{4g^2} + \frac{a}{2g}\right)$$

$$\therefore \quad R = \frac{v^2}{g}\left(1 - \tfrac{a^2}{g^2}\right) + \frac{2v^2a}{g^2}\,\frac{1}{2}\left(1 - \tfrac{a^2}{4g^2} + \tfrac{a}{2g}\right)^2$$

$$R = \frac{v^2}{g}\left(1 - \tfrac{a^2}{g^2}\right) + \frac{v^2a}{g^2}\left(1 - \tfrac{a^2}{4g^2} + \tfrac{a}{2g}\right)$$

$$= \frac{v^2}{g} + \frac{v^2a}{g^2} - \frac{v^2a^2}{g^3} - \frac{v^2a^3}{4g^4} + \frac{v^2a^2}{2g^3}$$

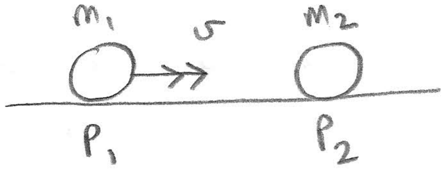
$$= \frac{v^2}{g} + \frac{v^2a}{g^2} - \frac{v^2a^2}{2g^3} - \frac{v^2a^3}{4g^4}$$

which justifies the required result

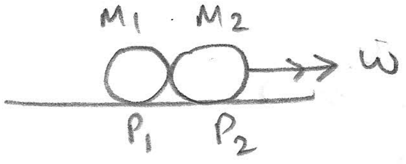
$$\boxed{R \cong \frac{v^2}{g} + \frac{v^2a}{g^2}}$$

Q9 (3)

10/



Before



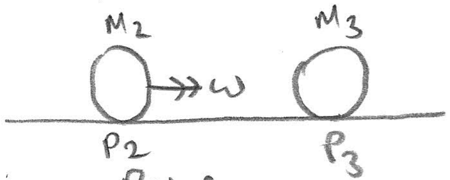
After

COLLISION 1

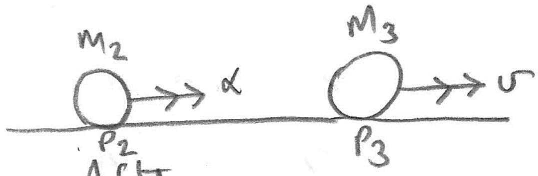
1st P1 Stationary

By conservation of momentum: $M_1 v = M_2 w \Rightarrow \boxed{w = \dfrac{M_1 v}{M_2}}$

Then.... COLLISION 2



Before



After

By conservation of momentum: $M_2 w = M_2 \alpha + M_3 v$

$$\therefore \quad \alpha = \frac{M_2\left(\frac{M_1 v}{M_2}\right) - M_3 v}{M_2}$$

$$\alpha = \frac{M_1 v}{M_2} - \frac{M_3 v}{M_2} \Rightarrow \boxed{\alpha = \frac{(M_1 - M_3) v}{M_2}}$$

Coefficient of restitution for Collision 1

$$e = \frac{w}{v} \quad \therefore \boxed{e = \frac{M_1}{M_2}}$$

Coefficient of restitution for Collision 2

$$e' = \frac{v - \alpha}{w} = \frac{v - \frac{(M_1 - M_3) v}{M_2}}{\frac{M_1 v}{M_2}}$$

$$= \boxed{\frac{M_2 - M_1 + M_3}{M_1}}$$

Q10(1)

Now for collisions to be possible *

$e' \geq 0$

$e' \leq 1$

$e' = 0$ inelastic
$e' = 1$ elastic.

[unless an explosion occurs!]

So since M_1, M_2, M_3 are ≥ 0

$e' \geq 0 \Rightarrow M_2 + M_3 \geq M_1$

$e' \leq 1 \Rightarrow M_2 + M_3 - M_1 \leq M_1$

$\Rightarrow M_2 + M_3 \leq 2M_1$

So $\boxed{2M_1 \geq M_2 + M_3 \geq M_1}$

The final energy of the system is

$$E = \tfrac{1}{2} M_2 \alpha^2 + \tfrac{1}{2} M_3 v^2$$

$$E = \tfrac{1}{2} M_2 \frac{(M_1 - M_3)^2 v^2}{M_2^2} + \tfrac{1}{2} M_3 v^2$$

$$\boxed{E = \tfrac{1}{2} v^2 \left[\frac{(M_1 - M_3)^2}{M_2} + M_3 \right]}$$

Now largest M_2 will minimize E and smallest M_2 will maximize E if $\boxed{M_1 \text{ and } M_3 \text{ are constant}}$

This is what the question says. [v also].

Now: $2M_1 \geq M_2 + M_3 \geq M_1$

$$\Rightarrow \boxed{\begin{matrix} M_2 \leq 2M_1 - M_3 \\ M_2 \geq M_1 - M_3 \end{matrix}}$$

Q10 (2)

But also $e \geq 0$ and $e \leq 1$

$\Rightarrow$ $M_1/M_2 \geq 0$ and $M_1/M_2 \leq 1$

Therefore $\boxed{M_2 \geq M_1}$ also

Hence the possible M_2 value are

$$\boxed{\begin{array}{l} M_2 \leq 2M_1 - M_3 \\ M_2 \geq M_1 \end{array}}$$

Since $M_1 - M_3 < M_1$ (!)

$$\therefore E_{max} = \frac{1}{2}v^2\left(\frac{(M_1-M_3)^2}{M_1} + M_3\right) \quad \text{if } M_2 = M_1$$

$$\boxed{E_{max} = \frac{1}{2}v^2\left(\frac{M_1^2 + M_3^2 - M_1M_3}{M_1}\right)}$$

$$\therefore E_{min} = \frac{1}{2}v^2\left(\frac{(M_1-M_3)^2}{2M_1-M_3} + M_3\right) \quad \text{if } M_2 = 2M_1 - M_3$$

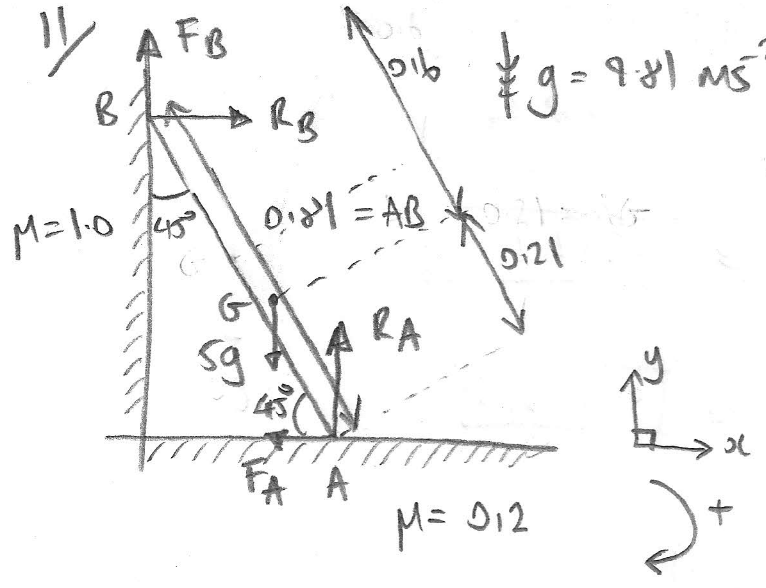
$$= \frac{1}{2}v^2\left(\frac{M_1^2 + \cancel{M_3^2} - \cancel{2M_1M_3} + \cancel{2M_1M_3} - \cancel{M_3^2}}{2M_1 - M_3}\right)$$

$$= \frac{1}{2}v^2 M_1^2 / 2M_1 - M_3$$

$$\therefore \boxed{E_{min} = \frac{M_1^2 v^2}{2(2M_1 - M_3)}}$$

Q10(3)

11/



$g = 9.81\ ms^{-2}$

Mass 5kg

System (a rod leaning against a rough wall and on a slightly less rough floor) is in equilibrium

Newton II: $\quad //x: \quad 0 = -F_A + R_B \quad ①$

$//y: \quad 0 = F_B - 5g + R_A \quad ②$

Taking moments $\circlearrowright +$

About A: $\quad 0 = -5g \times 0.21\cos 45° + F_B \times 0.81\cos 45° + R_B \times 0.81\sin 45° \quad ③$

About B: $\quad 0 = 5g \times 0.6\sin 45° + F_A \times 0.81\cos 45° - R_A \times 0.81\sin 45° \quad ④$

① $\quad R_B = F_A$

② $\quad R_A = 5g - F_B$

③ $\quad 1.05g = 0.81(F_B + R_B)$

④ $\quad 3g = 0.81(-F_A + R_A)$

[Since $\sin 45° = \cos 45°$] $= 1/\sqrt{2}$

In ④: $3g = 0.81(-R_B + 5g - F_B)$

ie ③ $\quad 0.81(R_B + F_B) = 1.05g$ ← So actually we don't have four independent eqn and can't solve for F_A, R_A, F_B, R_B.

let's use: $\quad R_B = F_A \quad ①$

$R_A = 5g - F_B \quad ②$

$\frac{3g}{0.81} = R_A - F_A \quad ④$

$\boxed{Q11\ ①}$

If limiting friction at A $\quad F_A = 0.2 R_A$

$\therefore$ (4) : $\frac{3g}{0.81} = R_A - 0.2 R_A = 0.8 R_A$

$\therefore \quad R_A = \frac{3g}{0.81 \times 0.8} \approx \boxed{45.42 \text{ N}}$

$\therefore \quad F_A = \frac{3g \times 0.2}{0.81 \times 0.8} \approx \boxed{9.083 \text{ N}}$

$\therefore$ In (1) $\quad R_B = F_A = \boxed{9.083 \text{ N}}$

In (2) $\quad F_B = 5g - R_A = 5g - \frac{3g}{0.81 \times 0.8} = \boxed{3.633 \text{ N}}$

$F_B / R_B = 0.3999\ldots$ $\qquad$ limiting friction at B means $F_B / R_B = 1.0$

$\boxed{\text{So if limiting at A} \Rightarrow \text{not limiting at B}}$

If limiting friction at B $\quad F_B = R_B \quad$ Now (1) $\Rightarrow F_A = R_B$

$\therefore$ In (2) : $R_A = 5g - R_B = 5g - F_A$

$\therefore \quad R_A + F_A = 5g \quad$ (5)

Now $\quad \frac{3g}{0.81} = R_A - F_A \quad$ (4)

$\therefore$ (5) + (4) $\quad 2R_A = 5g + 3g/0.81 \quad \Rightarrow R_A = \boxed{42.69 \text{ N}}$

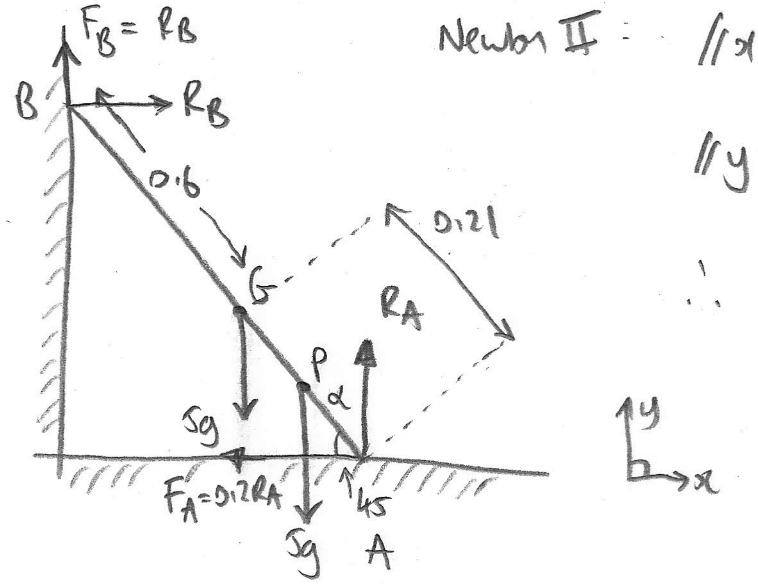
In (5) : $F_A = 5g - R_A = \boxed{6.358 \text{ N}}$

So $F_A / R_A = 0.149\ldots$ $\qquad$ limiting friction at A means $F_A / R_A = 0.2$.

$\boxed{\text{So if limiting at B} \Rightarrow \text{not limiting at A}}$

Q11 (2)

Now if a 5kg mass is attached at P s.t friction is limiting at BOTH A and B i.e. $\boxed{F_B = R_B \text{ and } F_A = 0.2R_A}$



Newton II : $\parallel x$: $0 = -0.2R_A + R_B$ ①

$\parallel y$: $0 = R_B - 10g + R_A$ ②

$\therefore$ ① − ②: $0 = 10g - 1.2R_A$

$\therefore R_A = 10g/1.2 = \boxed{81.75N}$

$\therefore R_B = 0.2 \times 10g/1.2 = \boxed{16.35N}$

[let $AP = \alpha$]

Taking moments about A ↻+

$$0 = -5g\,\alpha\cos45^\circ - 5g \times 0.21\cos45^\circ + R_B \times 0.81\cos45^\circ + F_B \times 0.81\cos45^\circ$$

$$\Rightarrow \quad 5g(\alpha + 0.21) = 0.81R_B \times 2 \quad \text{Since } F_B = R_B$$

$$\therefore \quad \alpha = \frac{0.81 \times 2 \times R_B}{5g} - 0.21$$

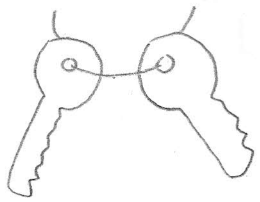
$$= \frac{0.81 \times 2 \times 0.2 \times 10g}{5g \times 1.2} - 0.21$$

$$= 0.54 - 0.21$$

$$\boxed{\alpha = 0.33}$$

Q11 ③

12/



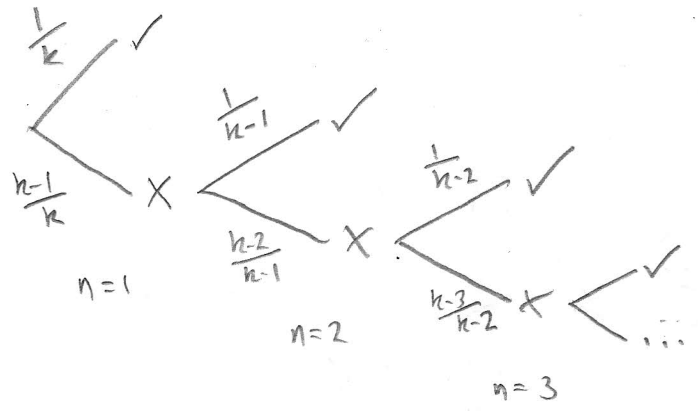
k keys on a key ring .... only one fits the front door.

Let $P(n,k)$ be the probability of finding the correct key from k after n attempts.

✓ means key matches lock

X means key doesn't fit lock

(i) Strategy # 1 : I select a key that I have not tried before, but otherwise all choices are equally likely



$$\therefore P(n,k) = \begin{cases} \frac{1}{k} & n = 1 \\ \underbrace{\frac{\cancel{(k-1)}\cancel{(k-2)}\cancel{(k-3)}\dots(k-n+1)}{(k)\cancel{(k-1)}\cancel{(k-2)}\dots(k-n+2)}}_{n-1\ X} \times \underbrace{\frac{1}{k-n+1}}_{\checkmark \text{ on } n} \end{cases}$$

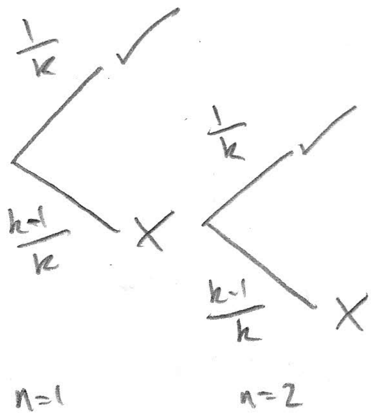
The more general case is also $\frac{1}{k}$

Since $\frac{(k-1)(k-2)\dots(k-n+2)(k-n+1)}{(k)(k-1)(k-2)\dots(k-n+2)(k-n+1)}$

is clearly $\frac{1}{k}$

So (i) $\boxed{P(n,k) = \frac{1}{k}}$

(ii) If I choose a key at random each time, without memory of which keys I have already tried, probabilities of ✓ and X are the same for each trial

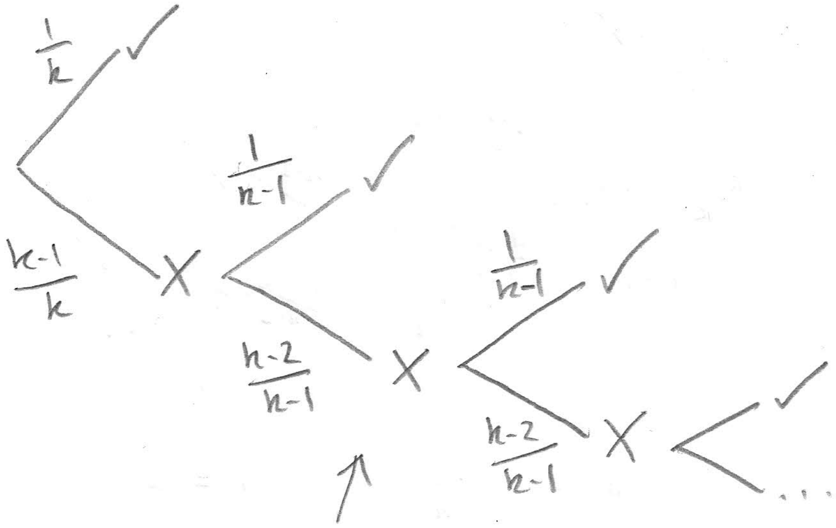


$$\therefore \boxed{P(n,k) = \left(\frac{k-1}{k}\right)^{n-1} \frac{1}{k}}$$

ie $P(n-1\ X \text{ then } \checkmark)$

Q12 (i)

(iii) At each attempt I choose from all keys apart from the previous key – assuming this was not successful.



If there subsequent branches are the same odds since only the previous key is excluded

$$\therefore \quad P(n,k) = \begin{cases} 1/k & n=1 \\ \underbrace{\left(\frac{k-1}{k}\right)\left(\frac{k-2}{k-1}\right)^{n-2}}_{n-1 \ \times} \underbrace{\left(\frac{1}{k-1}\right)}_{\checkmark \text{ on } n\text{th attempt}} & \end{cases}$$

$$\boxed{P(n,k) = \begin{cases} 1/k & n=1 \\ \frac{1}{k}\left(\frac{k-2}{k-1}\right)^{n-2} & n \neq 1 \end{cases}}$$

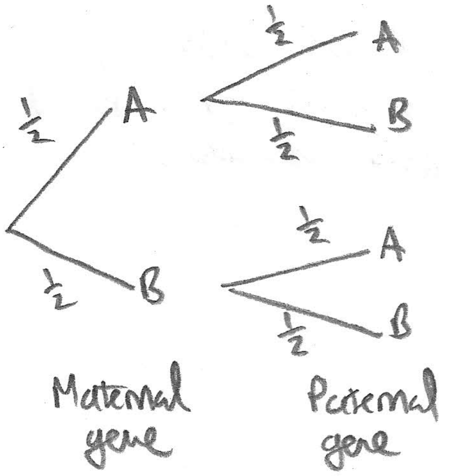
Clearly (i) is the better strategy!

$\dfrac{P(n,k)}{1/k}$	1	$\left(\frac{k-1}{k}\right)^{n-1}$	$\left(\frac{k-2}{k-1}\right)^{n-2}$
	(i)	(ii)	(iii)

Q12 (2)

13/

(i)



i.e one gene from each parent

← Assume equal probability of passing each gene since $\boxed{\text{both parents are AB}}$

Probability of an AA child is $P(AA) = \frac{1}{4}$

Probability of a BB is also $P(BB) = \frac{1}{4}$. Note $P(AB) = P(BA) = \frac{1}{4}$

So, if parents have $\boxed{\text{four}}$ children, desire $P(2 \text{ of } AA \ \& \ 2 \text{ of } BB)$

If all births are independent, the probability is

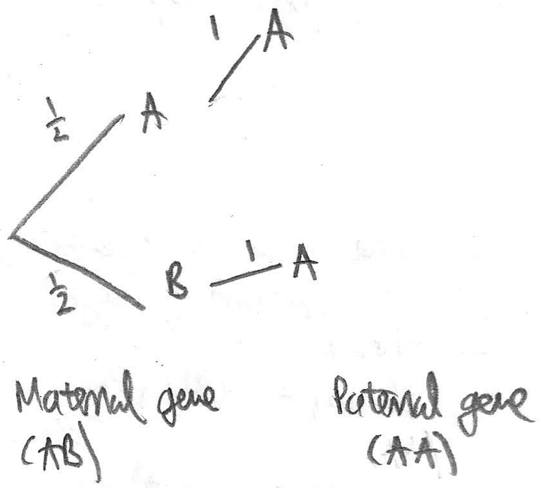
$$\frac{4!}{2!2!} \times P(AA)^2 \times P(BB)^2 = 6 \times \frac{1}{4^4} = \frac{6}{256} = \boxed{\frac{3}{128}}$$

↑

permutations of AA, AA, BB, BB

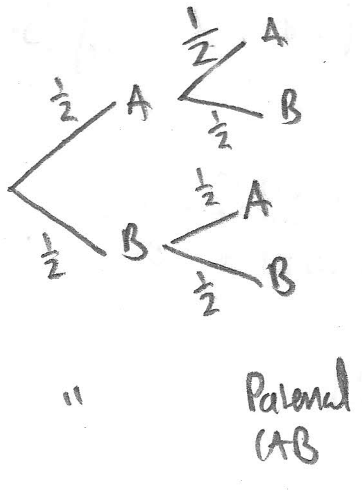
(ii) let mother be AB
let me be AA.

$\boxed{\text{Desire } P(AB)}$



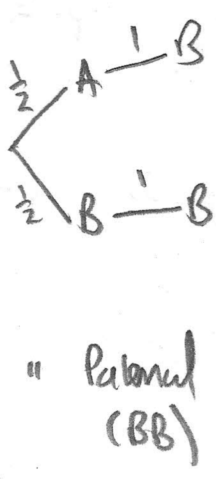
$\boxed{P(AA) = \frac{1}{2}}$

GIVEN Paternal AA



$\boxed{P(AA) = \frac{1}{4}}$

Given Paternal AB

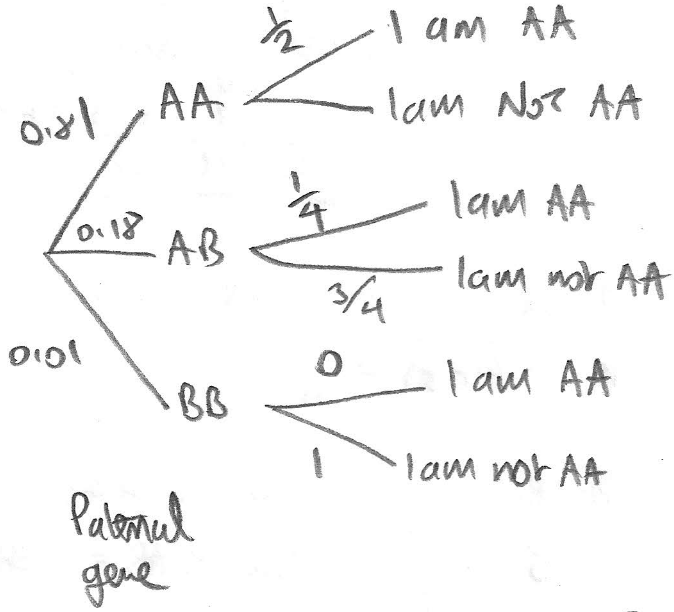


$\boxed{P(AA) = 0}$

Given Paternal BB

Q13①

So, assuming Mother is AB, population stats give probabilities for Paternal gene combinations



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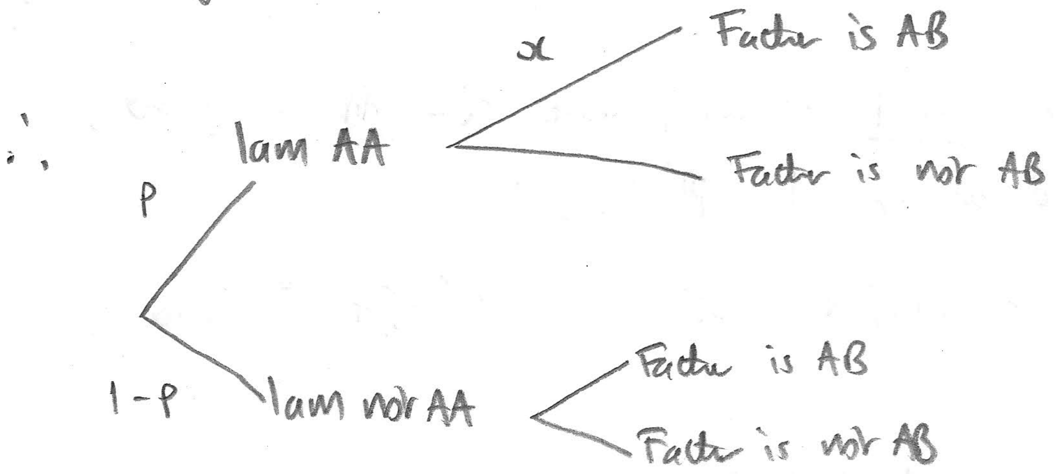

x is desired ie $x = P(\text{Father is AB} \mid \text{I am AA})$

Now $px = (0.18)(\tfrac{1}{4})$

$p = (\tfrac{1}{2})(0.81) + (\tfrac{1}{4})(0.18)$ (from top diagram)

So $x = \dfrac{0.18 \times 0.25}{0.5 \times 0.81 + 0.25 \times 0.18}$

$= \boxed{0.1}$

[Note this is essentially an application of Bayes' theorem

$P(A|B) = \dfrac{P(B|A)\,P(A)}{P(B)}$

$P(A|B) = \dfrac{P(B|A)\,P(A)}{P(B|A)P(A) + P(B'|A)P(A)}$

.... but drawing the tree diagrams adds to clarity!]

Q13 ②

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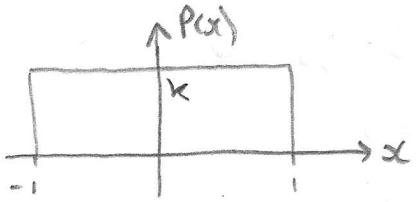

Probability density function for random variable x. ie $x \sim U(-1, 1)$
↑ uniform distribution

$$E[x] = \int_{-1}^{1} x\, p(x)\, dx = 0 \quad \text{by symmetry.}$$

Alternatively: $\displaystyle\int_{-1}^{1} xk\, dx = \left[\tfrac{1}{2}x^2 k\right]_{-1}^{1} = \frac{k}{2}\left[(1) - (1)\right] = 0$ ✓

Now $\displaystyle\int_{-1}^{1} p(x)\, dx = 1 \quad \therefore \quad k\left[x\right]_{-1}^{1} = 1 \Rightarrow 2k = 1$

$\Rightarrow \boxed{k = \tfrac{1}{2}}$

$$E[x^2] = \int_{-1}^{1} x^2 p(x)\, dx = \frac{1}{2}\left[\frac{1}{3}x^3\right]_{-1}^{1} = \boxed{\tfrac{1}{3}}$$

Variance $V[x] = E[x^2] - (E[x])^2$

$\therefore$ Since $E[x] = 0$, $V[x] = E[x^2] = \frac{1}{3}$

$$\therefore \quad V[x^2] = E[x^4] - (E[x^2])^2$$

$$= \int_{-1}^{1} x^4 \left(\tfrac{1}{2}\right) dx - \left(\tfrac{1}{3}\right)^2$$

$$= \frac{1}{2}\left[\frac{1}{5}x^5\right]_{-1}^{1} - \frac{1}{9}$$

$$= \frac{1}{10}(2) - \frac{1}{9} = \frac{1}{5} - \frac{1}{9} = \frac{9-5}{45} = \boxed{\tfrac{4}{45}}$$

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let $x, y \in U(-1,1)$ and define $z = y - x$ x is independent of y

$$E[z^2] = E[(y-x)^2]$$

$$= E[y^2 - 2xy + x^2]$$

$$= E[y^2] + E[x^2] - 2E[xy]$$

Now Covariance $\boxed{Cov[x,y] = E[xy] - E[x]\,E[y]}$

Q14 ①

If x and y are independent then $\text{Cov}[x,y] = 0$

Since $E[x] = E[y] = 0 \Rightarrow E[xy] = 0$

$\therefore E[z^2] = E[x^2] + E[y^2] = \boxed{2/3}$

$$V[z^2] = E[(y-x)^4] - (E[z^2])^2$$

$$= E[y^4 - 4y^3x + 6y^2x^2 - 4yx^3 + x^4] - \frac{4}{9}$$

Now since $\text{Cov}[x,y] = 0$, ie x,y independent

we can also state $E[x^n y^m] = E[x^n]\,E[y^m]$

$$\therefore V[z^2] = E[y^4] - 4E[y^3]E[x] + 6E[y^2]E[x^2] - 4E[y]E[x^3] + E[x^4] - 4/9$$

Also $\boxed{E[x^n] = E[y^n]}$

Since $x,y \sim U(-1,1)$

$E[x] = 0$

$E[x^2] = 1/3$

$E[x^3] = \int_{-1}^{1} \frac{1}{2}x^3\,dx = \frac{1}{2}\frac{1}{4}[x^4]_{-1}^{1} = 0$

{ x^3 odd symmetric interval so obvious! }

$E[x^4] = \int_{-1}^{1} \frac{1}{2}x^4\,dx = \frac{1}{2}\frac{1}{5}[x^5]_{-1}^{1} = \frac{1}{5}$

$$\therefore V[z^2] = \frac{1}{5} - 4(0)(0) + 6\frac{1}{3}\frac{1}{3} - 4(0)(0) + \frac{1}{5} - \frac{4}{9}$$

$$= \frac{2}{5} + \frac{2}{9}$$

$$= \frac{18+10}{45} = \boxed{28/45}$$

Recall $\boxed{V[x^2] = 4/45}$ from above.

$\therefore$ $\boxed{V[z^2] = 7\,V[x^2]}$ as required