

STEP I 2000

$$\frac{1}{\log_2 10} = \log_{10} 2 = 0.301029996$$

$$\log_{10} 3 = 0.477121255$$

(i)

Note $\log_{10} 5 = \log_{10} \left(\frac{10}{2}\right)$

$$= 1 - \log_{10} 2$$

$$= 0.698970004$$

$$= \boxed{0.699 \text{ to 3.d.p.}}$$

$$\log_{10} 6 = \log_{10}(3 \times 2)$$

$$= \log_{10} 2 + \log_{10} 3$$

$$= 0.778151251$$

$$= \boxed{0.778 \text{ to 3.d.p.}}$$

What are the first k significant digits of b^n ?

$$y = b^n$$

$$\log_{10} y = n \log_{10} b \quad \therefore \boxed{y = 10^{n \log_{10} b}}$$

Let $z = n \log_{10} b - \text{floor}(n \log_{10} b)$
i.e. the 'fractional part'.

First k digits are the first k digits of 10^{z+k-1}

This works since:

$$y = 10^z \times 10^{\text{floor}(n \log_{10} b)}$$

This just adds extra zeros!

So you could do this without a calculator

Now consider:

$$5 \times 10^{47} = 10^{\log_{10}(5 \times 10^{47})}$$

$$= 10^{\log_{10} 5 + 47}$$

$$\approx \boxed{10^{47.699}}$$

$$\left[\text{i.e. } x = 10^{\log_{10} x} \right]$$

$$3^{100} = 10^{\log_{10} 3^{100}}$$

$$= 10^{100 \log_{10} 3}$$

$$\approx \boxed{10^{47.712}}$$

$$\text{Hence } \boxed{5 \times 10^{47} < 3^{100} < 6 \times 10^{47}}$$

$$6 \times 10^{47} = 10^{\log_{10}(6 \times 10^{47})}$$

$$= 10^{\log_{10} 6 + 47}$$

$$\approx \boxed{10^{47.778}}$$

Q1(i)

(ii)

$$\text{let } y = 2^{1000}$$

$$y = 10^{\log_{10} 2^{1000}}$$

$$y = 10^{1000 \log_{10} 2}$$

$$1000 \log_{10} 2 = 301.029996 \dots$$

$$\text{so } y = 10^{0.02996 \dots} \times 10^{301}$$

$$y = 1.0715086 \dots \times 10^{301}$$

so first digit is $\boxed{1}$

$$\text{let } y = 2^{10,000}$$

$$y = 10^{\log_{10} 2^{10,000}}$$

$$y = 10^{10,000 \log_{10} 2}$$

$$10,000 \log_{10} 2 = 3010.29996 \dots$$

$$\text{so } y = 10^{0.29996 \dots} \times 10^{3010}$$

$$y = 1.99506312 \dots \times 10^{3010}$$

so first digit is $\boxed{1}$

$$\text{let } y = 2^{100,000}, \quad y = 10^{\log_{10} 2^{100,000}}, \quad y = 10^{100,000 \log_{10} 2}$$

$$100,000 \log_{10} 2 = 30,102.9996 \dots$$

$$\text{so } y = 10^{0.9996 \dots} \times 10^{30,102}$$

$$y = 9.990029 \dots \times 10^{30,102}$$

so first digit is $\boxed{9}$

2/ Consider $y = \left(x^4 - \frac{1}{x^2}\right)^5 \left(x - \frac{1}{x}\right)^6$

This will be a polynomial $y = x^{26} + \dots + x^{-16}$

To find the coefficient of x^{-12} we need to find n, m

s.t. $\left| \left(x^4\right)^n \left(-\frac{1}{x^2}\right)^{5-n} \left(x\right)^m \left(-\frac{1}{x}\right)^{6-m} \right| = x^{-12}$

$$4n + 2n - 10 + m + m - 6 = -12$$

$$6n + 2m = 4$$

$$\boxed{3n + m = 2}$$

Also $0 \leq n \leq 5$
 $0 \leq m \leq 6$

only solution is $n=0, m=2$

x^{-12} term is $\binom{5}{0} \left(x^4\right)^0 \left(-\frac{1}{x^2}\right)^5 \times \binom{6}{2} \left(x\right)^2 \left(-\frac{1}{x}\right)^4$
 $= -x^{-10} \times 15 x^2 x^{-4} = \boxed{-15x^{-12}}$

$$\left[\binom{6}{2} = \frac{6!}{4!2!} = \frac{6 \times 5 \times 4!}{4!2!} = 3 \times 5 = 15 \right]$$

Similarly, to find the coefficient of x^2 we need to find n, m

s.t. $\left| \left(x^4\right)^n \left(-\frac{1}{x^2}\right)^{5-n} \left(x\right)^m \left(-\frac{1}{x}\right)^{6-m} \right| = x^2$

$0 \leq n \leq 5$
 $0 \leq m \leq 6$

$$4n + 2n - 10 + m + m - 6 = 2$$

$$6n + 2m = 18$$

$$\boxed{3n + m = 9}$$

Solutions are:

$$m=0, n=3$$

$$m=6, n=1$$

$$m=3, n=2$$

Now coefficient of term $x^{6n+2m-16}$ is

$$(-1)^{5-n+6-m} \binom{5}{n} \binom{6}{m}$$

