8/ $P(x) = Q(x)R'(x) - Q'(x)R(x)$

$\therefore \dfrac{P(x)}{Q^2(x)} = \dfrac{d}{dx}\left(\dfrac{R}{Q}\right)$

{Quotient rule}
$\left\{ \dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \right\}$

So $\displaystyle\int \frac{P(x)}{(Q(x))^2}\,dx = \boxed{\frac{R}{Q} + k}$ ← (k constant)

(i). Consider $I = \displaystyle\int \frac{5x^2 - 4x - 3}{(1+2x+3x^2)^2}\,dx$

So $Q(x) = 1 + 2x + 3x^2$

let $R(x) = a + bx + cx^2$ as suggested

$P(x) = 5x^2 - 4x - 3$

$= \underbrace{(1+2x+3x^2)}_{Q(x)} \times \underbrace{(b+2cx)}_{R'(x)} - \underbrace{(2+6x)}_{Q'(x)} \times \underbrace{(a+bx+cx^2)}_{R(x)}$

$\therefore 5x^2 - 4x - 3 = x^2(3b + 4c - 2c - 6b)$

$+ x(2c + 2b - 2b - 6a)$

$+ b - 2a$

Comparing coefficients of x:

$x^2: \quad 5 = 2c - 3b \quad$ ①

$x^1: \quad -4 = 2c - 6a \quad$ ②

$x^0: \quad -3 = b - 2a \quad$ ③

① - ② $\Rightarrow 9 = -3b + 6a$
$\Rightarrow \boxed{-3 = b - 2a}$

So a unique solution is <u>not possible</u>

as ①, ② yield ③ ie we don't have three independent equations

let $\boxed{a = 0}$: ③ $\Rightarrow \boxed{b = 2a - 3} \Rightarrow \boxed{b = -3}$

② $\Rightarrow \boxed{c = 3a - 2} \Rightarrow \boxed{c = -2}$

(24)

let $\boxed{a=1:}$ $\quad b = 2a-3 \Rightarrow \boxed{b = -1}$

$c = 3a-2 \Rightarrow \boxed{c = 1}$

lets consider both cases for I

$\boxed{a=0}$ $\quad I = \displaystyle\int \frac{5x^2-4x-3}{(1+2x+3x^2)^2}\,dx = \frac{R}{Q} + k$

$R = a + bx + cx^2$

$\therefore R(x) = -3x - 2x^2$

$$\therefore \boxed{I = \frac{-3x-2x^2}{1+2x+3x^2} + k}$$

$\boxed{a=1}$ $\quad R(x) = 1 - x + x^2$

$$\therefore \frac{R}{Q} = \frac{1-x+x^2}{1+2x+3x^2}$$

Now $1 + \dfrac{-3x-2x^2}{1+2x+3x^2} = \dfrac{1+2x+3x^2-3x-2x^2}{1+2x+3x^2}$

$$= \frac{1-x+x^2}{1+2x+3x^2}$$

So if we absorb the extra 1 into the integration constant, both yield the same function of x.

In terms of a: $\quad R(x) = a + (2a-3)x + (3a-2)x^2$

So $\boxed{I = \dfrac{a+(2a-3)x+(3a-2)x^2}{1+2x+3x^2} + k}$ Any a is allowed!

Note $I(0) = a + k$

(ii) Consider $(1+\cos x + 2\sin x)\frac{dy}{dx} + (\sin x - 2\cos x)y = 5 - 3\cos x + 4\sin x$

ie $\frac{dy}{dx} + \frac{\sin x - 2\cos x}{1+\cos x + 2\sin x}y = \frac{5-3\cos x + 4\sin x}{1+\cos x + 2\sin x}$ (*)

Now ODE'S of the form $\frac{dy}{dx} + f(x)y = g(x)$

can be made separable via use of an integrating factor $e^{\int f(x)\,dx}$

$$e^{\int f(x)dx}\frac{dy}{dx} + e^{\int f(x)dx}f(x)y = e^{\int f(x)dx}g(x)$$

$$\frac{d}{dx}\left(e^{\int f(x)dx}y\right) = e^{\int f(x)dx}\frac{dy}{dx} + e^{\int f(x)dx}f(x)\,y$$

$$\therefore \frac{d}{dx}\left(e^{\int f(x)dx}y\right) = e^{\int f(x)dx}g(x)$$

$$\therefore e^{\int f(x)dx}y = \int e^{\int f(x)dx}g(x)\,dx$$

or $$\boxed{y = \frac{\int e^{\int f(x)dx}g(x)\,dx}{e^{\int f(x)dx}}}$$

In our case (*) $f(x) = \frac{\sin x - 2\cos x}{1+\cos x + 2\sin x}$ $\frac{d}{dx}$

$\int f(x)\,dx = -\ln|1+\cos x + 2\sin x| + c$ Since $\int \frac{h'(x)}{h(x)}\,dx = \ln|h(x)| + C$

Ignore c in IF as multiply both sides of equation.

Since the $|\ldots|$ is always >0

$$\therefore\quad e^{\int f(x)\,dx} = e^{-\ln(1+\cos x+2\sin x)}$$

$$= \frac{1}{1+\cos x+2\sin x}$$

$$\therefore\quad y = \frac{\displaystyle\int \frac{5-3\cos x+4\sin x}{(1+\cos x+2\sin x)^2}\,dx}{\dfrac{1}{1+\cos x+2\sin x}}$$

ie $$y = (1+\cos x+2\sin x)\int \underbrace{\frac{5-3\cos x+4\sin x}{(1+\cos x+2\sin x)^2}}\,dx$$

By analogy to part (i) $\quad \displaystyle\int \frac{P(x)}{Q^2(x)}\,dx$

$P(x) = 5-3\cos x+4\sin x$

$Q(x) = 1+\cos x+2\sin x$

let $\boxed{R(x) = a+b\sin x+c\cos x}$

$\therefore$ if $P(x) = QR' - Q'R$

$$5-3\cos x+4\sin x = (1+\cos x+2\sin x)(b\cos x - c\sin x) - (-\sin x+2\cos x)(a+b\sin x+c\cos x)$$

$$5-3\cos x+4\sin x = b\cos x - c\sin x + b\cos^2 x - c\sin x\cos x + 2b\sin x\cos x - 2c\sin^2 x + \sin x(a+b\sin x+c\cos x) - 2\cos x(a+b\sin x+c\cos x)$$

$$= b\cos x - c\sin x + b\cos^2 x - \cancel{c\sin x\cos x} + \cancel{2b\sin x\cos x} - 2c\sin^2 x + a\sin x + b\sin^2 x + \cancel{c\sin x\cos x} - 2a\cos x - \cancel{2b\sin x\cos x} - 2c\cos^2 x$$

$$\therefore\ 5 - 3\cos x + 4\sin x = b(\cos^2 x + \sin^2 x) - 2c(\cos^2 x + \sin^2 x)$$
$$+ \cos x(b - 2a) + \sin x(-c + a)$$

$$\therefore\ 5 - 3\cos x + 4\sin x = b - 2c + (b-2a)\cos x + (a-c)\sin x$$

comparing coefficients

$$5 = b - 2c \quad (1)$$
$$-3 = b - 2a \quad (2)$$
$$4 = a - c \quad (3)$$

$$(1) - (2):\quad 8 = -2c + 2a$$
$$4 = a - c \quad \text{ie } (3)$$

So, as above, these are not independent. In terms of a:

$$b = 2a - 3$$
$$c = a - 4$$

$$\therefore\ R(x) = a + (2a-3)\sin x + (a-4)\cos x$$

$$\therefore\ \int \frac{P(x)}{Q^2(x)}\,dx = \frac{R}{Q} + k$$

$$\Rightarrow\quad y = (1 + \cos x + 2\sin x)\left(\frac{a + (2a-3)\sin x + (a-4)\cos x}{1 + \cos x + 2\sin x} + k\right)$$

$$\text{i.e}\quad \boxed{y = a + (2a-3)\sin x + (a-4)\cos x + k(1 + \cos x + 2\sin x)}$$

$$\text{If } a = 4:\quad \boxed{y = 4 + 5\sin x + k(1 + \cos x + 2\sin x)}$$