4/ (i)

$x^2 + ax + b = 0$ ①

$x^2 + cx + d = 0$ ②

if α is a common solution

① − ② $(a-c)\alpha + b - d = 0$

$$\boxed{\alpha = \frac{-(b-d)}{a-c}} \qquad a \neq c \qquad (*)$$

Substituting the above result (*) into ①

$$\frac{(b-d)^2}{(a-c)^2} - \frac{a(b-d)}{a-c} + b = 0$$

$$(+) \quad \boxed{(b-d)^2 - a(b-d)(a-c) + b(a-c)^2 = 0}$$

← which is the condition specified

Substituting (*) into ②

$$\frac{(b-d)^2}{(a-c)^2} - \frac{c(b-d)}{a-c} + d = 0$$

$$\boxed{(b-d)^2 - c(b-d)(a-c) + d(a-c)^2 = 0} \qquad (++)$$

(++) − (+)

$$(-a+c)(b-d)(a-c) + (b-d)(a-c)^2 = 0$$

$$-(b-d)(a-c)^2 + (b-d)(a-c)^2 = 0$$

So consistent.
(is this really necessary? Surely (+) is enough, since (*) comes from ① and ②)

Now if $a = c$ (+) becomes

$(b-d)^2 = 0$ ie $b = d$

So ①, ② are $x^2 + ax + b = 0$ ①

$x^2 + ax + b = 0$ ②

ie same equation, so common solution is that of ①

(12)

ie $\alpha = \dfrac{-a \pm \sqrt{a^2-4(1)b}}{2}$

$$\boxed{\alpha = -\tfrac{1}{2}a \pm \tfrac{1}{2}\sqrt{a^2-4b}}$$

(ii) Now consider $x^2+ax+b=0$ ①

$x^3+(a+1)x^2+qx+r=0$ ②

Part (i) summary: $(b-d)^2 - a(b-d)(a-c) + b(a-c)^2 = 0$

$\Rightarrow$ $\left.\begin{array}{l} x^2+ax+b=0 \\ x^2+cx+d=0 \end{array}\right\}$ have a common root

So if $\boxed{(b-r)^2 - a(b-r)(a+b-q) + b(a+b-q)^2 = 0}$ (*)

this means (making the correspondences $d \leftrightarrow r$, $c \leftrightarrow q-b$)

$$\boxed{\begin{array}{l} x^2+ax+b=0 \\ x^2+(q-b)x+r=0 \end{array}}$$ have a common root.

Write ① as $x^3+ax^2+bx=0$ ③

$\therefore$ $x^3+(a+1)x^2+qx+r - x^3-ax^2-bx = 0$ ② − ③

$\Rightarrow$ $\boxed{x^2+x(q-b)+r=0}$

So everything is consistent. $\therefore$ (*) can be used to find solutions of ①, ②

Consider ①: $2x^2+5x+2b=0 \Rightarrow x^2+\frac{5}{2}x+b=0$

②: $2x^3+7x^2+5x+1=0 \Rightarrow x^3+\frac{7}{2}x^2+\frac{5}{2}x+\frac{1}{2}=0$

Using the formula above

$a = \frac{5}{2}$ $\qquad$ ($a+1$ is indeed $\frac{7}{2}$ ✓)

$q = \frac{5}{2}$

$r = \frac{1}{2}$

$\therefore$ Substituting into (*)

$$(b-\tfrac{1}{2})^2 - \tfrac{5}{2}(b-\tfrac{1}{2})(\tfrac{5}{2}+b-\tfrac{5}{2}) + b(\tfrac{5}{2}+b-\tfrac{5}{2})^2 = 0$$

$$(b-\tfrac{1}{2})^2 - \tfrac{5}{2}(b-\tfrac{1}{2})b + b^3 = 0$$

$$2b^3 - 5b(b-\tfrac{1}{2}) + 2(b-\tfrac{1}{2})^2 = 0$$

$$4b^3 - 5b(2b-1) + (2b-1)^2 = 0$$

$$4b^3 - 10b^2 + 5b + 4b^2 - 4b + 1 = 0$$

$$4b^3 - 6b^2 + b + 1 = 0 \qquad (\dagger)$$

$b = 1$: $\quad 4 - 6 + 1 + 1 = 0$ $\qquad$ so $(b-1)$ is a factor of $(\dagger)$

$$
\begin{array}{l}
\qquad\quad 4b^2 - 2b - 1 \\
b-1 \,\big)\, 4b^3 - 6b^2 + b + 1 \\
\quad -4b^2(b-1) = -4b^3 + 4b^2 \\
\hline
\quad -2b^2 + b + 1 \\
+2b(b-1) = +2b^2 - 2b \\
\hline
\quad -b + 1 \\
+(b-1) = +b - 1 \\
\hline
\quad 0
\end{array}
$$

$\therefore 4b^3 - 6b^2 + b + 1$

$= (b-1)(4b^2 - 2b - 1)$

other roots are when

$4b^2 - 2b - 1 = 0$

$\Rightarrow b = \dfrac{2 \pm \sqrt{4 - 4(4)(-1)}}{8}$

$$\boxed{b = \frac{1 \pm \sqrt{5}}{4}}$$

is half the golden ratio!