13/ $\boxed{\mathrm{Corr}(U,V) = \dfrac{\mathrm{Cov}(U,V)}{\sqrt{V[U]V[V]}}}$ $\qquad V[U] \equiv \mathrm{Var}[U]$

Product Moment correlation coefficient

independent random variables $\boxed{Z_1, Z_2, Z_2}$ have expectation 0 and variance 1

Note $\boxed{\mathrm{Cov}(U,V)} = E[(U-\bar{U})(V-\bar{V})]$ $\qquad \boxed{\begin{array}{l}\bar{U} = E[U] \\ \bar{V} = E[V]\end{array}}$

$= \boxed{E[UV] - \bar{U}\bar{V}}$

If U, V are independent then $\mathrm{Cov}(U,V) = 0$

So $\boxed{\mathrm{Corr}(Z_1 Z_2) = 0}$

let $\boxed{Y_1 = Z_1}$ $\qquad \boxed{Y_2 = \rho_{12} Z_1 + (1-\rho_{12}^2)^{\frac{1}{2}} Z_2}$

$\boxed{-1 < \rho_{12} < 1}$ $\qquad$ Note following results

$E[aX \pm bY] = aE[X] \pm bE[Y]$

$V[aX \pm bY] = a^2 V[X] + b^2 V[Y] \pm 2ab\,\mathrm{Cov}[X,Y]$

$\therefore E[Y_2] = \rho_{12} E[Z_1] + (1-\rho_{12}^2)^{\frac{1}{2}} E[Z_2]$

$= \boxed{0}$ $\qquad$ Since $E[Z_1] = 0$, $E[Z_2] = 0$

$V[Y_2] = \rho_{12}^2 V[Z_1] + (1-\rho_{12}^2) V[Z_2] + 2\rho_{12}(1-\rho_{12}^2)^{\frac{1}{2}} \mathrm{Cov}[Z_1 Z_2]$

Now $V[Z_1] = V[Z_2] = 1$ and $\mathrm{Cov}[Z_1 Z_2] = 0$.

$\therefore \boxed{V[Y_2] = 1}$

$\mathrm{Cov}[Y_1, Y_2] = E[Y_1 Y_2] - \underbrace{E[Y_1] \times E[Y_2]}_{\text{Both zero}} = E[Y_1 Y_2]$

$$Y_1 Y_2 = \rho_{12} Z_1^2 + (1-\rho_{12}^2)^{\frac{1}{2}} Z_1 Z_2$$

$$\therefore \quad E[Y_1 Y_2] = \rho_{12} E[Z_1^2] + (1-\rho_{12}^2) E[Z_1 Z_2]$$

Now $\quad Cov[Z_1 Z_2] = E[Z_1 Z_2] - E[Z_1]E[Z_2]$

$\quad 0 = E[Z_1 Z_2] - 0 \times 0$

$\therefore \quad \boxed{E[Z_1 Z_2] = 0}$

$\therefore \quad E[Y_1 Y_2] = \rho_{12} E[Z_1^2]$

Now $\quad \underbrace{V[Z_1]}_{1} = E[Z_1^2] - (\underbrace{E[Z_1]}_{\text{zero}})^2$

$\therefore \quad \boxed{E[Z_1^2] = 1} \quad \therefore \quad E[Y_1 Y_2] = \rho_{12}.$

$\therefore \quad Cov[Y_1 Y_2] = \rho_{12}$

$\therefore \quad$ Since $V[Y_1] = 1$ and, from above, $V[Y_2] = 1$

$\Rightarrow \quad \boxed{Corr[Y_1, Y_2] = \rho_{12}}$

Now let $Y_3 = aZ_1 + bZ_2 + cZ_3 \qquad a, b, c \in \mathbb{R}^+$

$E[Y_3] = 0 \qquad V[Y_3] = 1 \qquad Corr(Y_1, Y_3) = \rho_{13} \qquad Corr(Y_2, Y_3) = \rho_{23}$

$Cov[Y_1, Y_3] = E[aZ_1^2 + bZ_2 Z_1 + cZ_3 Z_1] - E[Y_1]E[Y_3]$

$= a \quad$ Since $E[Z_1^2] = 1$ and all other terms are zero

$\therefore \quad$ Since $V[Y_1] V[Y_3] = 1 \Rightarrow Corr(Y_1, Y_3) = a \qquad \therefore \boxed{a = \rho_{13}}$

$$\text{Cov}[Y_2, Y_3] = E\left[\left(\rho_{12} Z_1 + (1-\rho_{12}^2)^{\frac{1}{2}} Z_2\right)(aZ_1 + bZ_2 + cZ_3)\right] - \underset{0}{E[Y_2]}\,\underset{0}{E[Y_3]}$$

$$= \rho_{12} a\, E[Z_1^2] + (1-\rho_{12}^2) b\, E[Z_2^2]$$

$$= \underset{a = \rho_{13}}{\rho_{12}\rho_{13}} + (1-\rho_{12}^2)^{\frac{1}{2}} b \qquad \text{Since} \quad E[Z_1^2] = E[Z_2^2] = 1$$

$\therefore$ Since $V[Y_3] = 1$ and $\text{Cov}(Y_2, Y_3) = \rho_{23}$

$$\Rightarrow \quad \rho_{23} = \rho_{12}\rho_{13} + (1-\rho_{12}^2)^{\frac{1}{2}} b$$

$$\therefore \quad \boxed{\frac{\rho_{23} - \rho_{12}\rho_{13}}{(1-\rho_{12}^2)^{\frac{1}{2}}} = b}$$

Now $V[Y_3] = 1$, and since $Z_1\ Z_2\ Z_3$ all independent

$$V[Y_3] = a^2 V[Z_1] + b^2 V[Z_2] + c^2 V[Z_3]$$

$$1 = a^2 + b^2 + c^2$$

$$\therefore \quad c = \sqrt{1 - a^2 - b^2}$$

$$\therefore \quad \boxed{c = \sqrt{1 - \rho_{13}^2 - \frac{(\rho_{23} - \rho_{12}\rho_{13})^2}{1-\rho_{12}^2}}}$$

Now let $\boxed{X_i = \mu_i + \sigma_i Y_i}$ $\quad [i = 1,2,3]$

ie
$$E[X_i] = \mu_i + \sigma_i E[Y_i] = \boxed{\mu_i}$$
$$V[X_i] = V[\mu_i + \sigma_i Y_i] = V[\sigma_i Y_i] = \sigma_i^2 V[Y_i] = \boxed{\sigma_i^2}$$

$$\mathrm{Corr}(X_i, X_j) = \frac{\mathrm{Cov}[X_i, X_j]}{\sqrt{V[X_i]\, V[X_j]}}$$

$$= \frac{\mathrm{Cov}[X_i, X_j]}{\sqrt{\sigma_i^2 \sigma_j^2}}$$

$$\mathrm{Cov}[X_i, X_j] = E[(\mu_i + \sigma_i Y_i)(\mu_j + \sigma_j Y_j)] - \mu_i \mu_j$$

$$= E[\mu_i \mu_j + \sigma_i \mu_j Y_i + \mu_i \sigma_j Y_j + \sigma_i \sigma_j Y_i Y_j] - \mu_i \mu_j$$

Since $E[Y_i] = 0 \quad E[Y_j] = 0$

$\therefore \quad \mathrm{Cov}[X_i, X_j] = \sigma_i \sigma_j E[Y_i Y_j]$

$\therefore \quad \mathrm{Corr}(X_i, X_j) = E[Y_i Y_j]$

Now since $V[Y_i], V[Y_j] = 1$, $E[Y_i] = 0$, $E[Y_j] = 0$

$E[Y_i Y_j] = \mathrm{Corr}(Y_i, Y_j)$

$\therefore \quad \boxed{\mathrm{Corr}(X_i, X_j) = \rho_{ij}}$

This is an example of a general idea

if $Y = A + By$ ie linear transformation
$X = a + bx$

$$\boxed{\mathrm{Corr}(X, Y) = \mathrm{Corr}(x, y)}$$

Fin!

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