STEP III 2010

1/ $\{x_1, x_2 \ldots\ldots x_{n+1}\}$ are any real numbers

$$A = \frac{1}{n}\sum_{k=1}^{n} x_k \qquad B = \frac{1}{n}\sum_{k=1}^{n}(x_k - A)^2$$

$$C = \frac{1}{n+1}\sum_{k=1}^{n+1} x_k \qquad D = \frac{1}{n+1}\sum_{k=1}^{n+1}(x_k - C)^2$$

(i) $$nA = \sum_{n=1}^{n} x_k \; ; \quad \sum_{k=1}^{n+1} x_k = \sum_{k=1}^{n} x_k + x_{n+1}$$

$$\therefore \sum_{k=1}^{n+1} x_k = nA + x_{n+1}$$

$$\therefore \boxed{C = \frac{nA + x_{n+1}}{n+1}}$$

(ii) $$B = \frac{1}{n}\sum_{k=1}^{n}\left(x_k^2 - 2x_k A + A^2\right)$$

$$B = \frac{1}{n}\sum_{k=1}^{n} x_k^2 - 2A\frac{1}{n}\sum_{k=1}^{n} x_k + \frac{nA^2}{n}$$

$$B = \frac{1}{n}\sum_{k=1}^{n} x_k^2 - 2A^2 + A^2$$

$$\boxed{B = \frac{1}{n}\sum_{k=1}^{n} x_k^2 - A^2} \Rightarrow \boxed{\sum_{k=1}^{n} x_k^2 = (B + A^2)n}$$

(iii) $$D = \frac{1}{n+1}\sum_{k=1}^{n+1}\left(x_k^2 - 2Cx_k + C^2\right)$$

$$D = \frac{1}{n+1}\left\{\sum_{k=1}^{n+1} x_k^2 - 2C\sum_{k=1}^{n+1} x_k + (n+1)C^2\right\}$$

① $$D = \frac{1}{n+1}\left\{\sum_{k=1}^{n} x_k^2 + x_{n+1}^2 - 2C^2(n+1) + (n+1)C^2\right\}$$

$$\boxed{D = \frac{1}{n+1}\left(n(B+A^2) + x_{n+1}^2\right) - C^2}$$

Now we desire an expression involving B, A, x_{n+1}, n

So using $\boxed{C = \frac{1}{n+1}(nA + x_{n+1})}$ $\{$ Goal is to show $(n+1)D \geq nB$ $\}$

$$\Rightarrow \quad D = \frac{1}{n+1}\left(n(B+A^2) + x_{n+1}^2\right) - \frac{1}{(n+1)^2}(nA + x_{n+1})^2$$

$$(n+1)D = n(B+A^2) + x_{n+1}^2 - \frac{(nA + x_{n+1})^2}{n+1}$$

$$(n+1)D = nB + nA^2 + x_{n+1}^2 - \frac{n^2A^2 + 2nAx_{n+1} + x_{n+1}^2}{n+1}$$

$$(n+1)D = \frac{nB(n+1) + nA^2(n+1) - n^2A^2 + (n+1)x_{n+1}^2 - 2nAx_{n+1} - x_{n+1}^2}{n+1}$$

$$(n+1)^2 D = n(n+1)B + \cancel{n^2A^2} + nA^2 - \cancel{n^2A^2} + nx_{n+1}^2 - 2nAx_{n+1}$$

$$(n+1)^2 D = n(n+1)B + nA^2 + nx_{n+1}^2 - 2nAx_{n+1}$$

$$\frac{(n+1)^2 D}{n} = (n+1)B + (x_{n+1} - A)^2$$

$$\therefore \quad (n+1)D = \frac{n}{n+1}\left\{(n+1)B + (x_{n+1} - A)^2\right\}$$

$$\Rightarrow \quad \boxed{(n+1)D = nB + \frac{n}{n+1}(x_{n+1} - A)^2}$$

Since the $(..)^2$ term must be positive, since $n \geq 1$

$$\Rightarrow \quad \boxed{(n+1)D \geq nB \quad \forall\, x_{n+1}}$$

For the last part we want to investigate $D < B$

$\therefore$ Consider $D - B$, or better (given above) $(D-B)(n+1)$

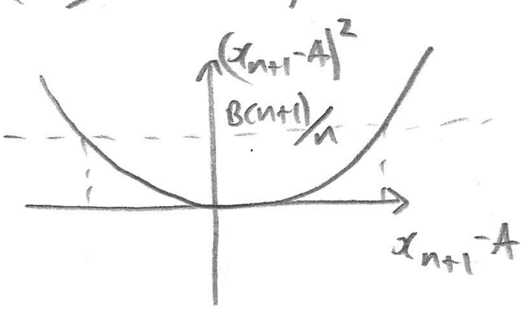
$$(D-B)(n+1) = D(n+1) - B(n+1)$$

$$= nB + \frac{n}{n+1}(x_{n+1} - A)^2 - B(n+1) = \boxed{\frac{n}{n+1}(x_{n+1}-A)^2 - B}$$

Note $B = \frac{1}{n}\sum_{k=1}^{n}(x_k - A)^2 \Rightarrow B \geq 0$

$D - B$ can only be negative (ie $D < B$)

if $\frac{n}{n+1}(x_{n+1} - A)^2 < B$



So since $B \geq 0$

$$(x_{n+1} - A)^2 < \frac{B(n+1)}{n}$$

$$\Rightarrow \quad x_{n+1} - A > -\sqrt{\frac{B(n+1)}{n}}$$

$$\Rightarrow \quad x_{n+1} > A - \sqrt{\frac{B(n+1)}{n}}$$

or

$$x_{n+1} - A < \sqrt{\frac{B(n+1)}{n}}$$

$$\Rightarrow \quad x_{n+1} < A + \sqrt{\frac{B(n+1)}{n}}$$

Hence $\boxed{A - \sqrt{\frac{B(n+1)}{n}} < x_{n+1} < A + \sqrt{\frac{B(n+1)}{n}}}$

as required.