a/ $\cosh a = \frac{1}{2}(e^a + e^{-a})$

(i) $$I = \int_0^1 \frac{1}{x^2+2x\cosh a+1}\,dx$$

$$= \int_0^1 \frac{1}{x^2+\cancel{2}x\frac{1}{\cancel{2}}(e^a+e^{-a})+1}\,dx$$

$$= \int_0^1 \frac{1}{x^2+xe^a+xe^{-a}+1}\,dx$$

$$= \int_0^1 \frac{1}{(x+e^a)(x+e^{-a})}\,dx \qquad [e^a e^{-a} = 1]$$

let $$\frac{1}{(x+e^a)(x+e^{-a})} = \frac{A}{x+e^a} + \frac{B}{x+e^{-a}}$$

$\therefore$ $1 = A(x+e^{-a}) + B(x+e^a)$

let $x = -e^a$: $1 = A(e^{-a}-e^a)$

$$\therefore \boxed{A = -\frac{1}{e^a-e^{-a}}}$$

let $x = -e^{-a}$: $1 = B(-e^{-a}+e^a)$

$$\therefore \boxed{B = \frac{1}{e^a-e^{-a}}}$$

$$\therefore I = \int_0^1 \left[\left(-\frac{1}{e^a-e^{-a}}\right)\frac{1}{x+e^a} + \left(\frac{1}{e^a-e^{-a}}\right)\frac{1}{x+e^{-a}}\right]dx$$

$$I = \left[-\frac{1}{e^a - e^{-a}} \ln|x+e^a| + \frac{1}{e^a - e^{-a}} \ln|x+e^{-a}| \right]_0^1$$

can ignore $|\dots|$ since $x \geq 0$ and $e^{\pm a} > 0$

$$\therefore \quad I = \frac{1}{e^a - e^{-a}} \left[\ln\left(\frac{x+e^{-a}}{x+e^a}\right) \right]_0^1$$

$$I = \frac{1}{e^a - e^{-a}} \left(\ln\left(\frac{1+e^{-a}}{1+e^a}\right) - \ln\left(e^{-2a}\right) \right)$$

$$\sinh a = \tfrac{1}{2}(e^a - e^{-a})$$

$$I = \frac{1}{2\sinh a} \ln\left(\frac{1+e^{-a}}{(1+e^a)e^{-2a}}\right)$$

$$= \frac{1}{2\sinh a} \ln\left(\frac{1+e^{-a}}{e^{-a}+1} \frac{1}{e^{-a}}\right)$$

$$= \frac{1}{2\sinh a} \ln(e^a)$$

$$\therefore \quad \boxed{I = \frac{a}{2\sinh a}} \quad \text{as required}$$

(ii) Now we will apply the same idea to two more integrals

$$I = \int_1^\infty \frac{1}{x^2 + 2x\sinh a - 1} \, dx$$

$$x^2 + 2x\sinh a - 1 = x^2 + xe^{a} - xe^{-a} - 1$$
$$= (x + e^{a})(x - e^{-a})$$

$$\therefore \quad \frac{1}{x^2 + 2x\sinh a - 1} = \frac{A}{x+e^{a}} + \frac{B}{x-e^{-a}}$$

$$\therefore \quad 1 = A(x-e^{-a}) + B(x+e^{a})$$

$x = e^{-a}$: $\quad 1 = B(e^{-a}+e^{a}) \quad \therefore \quad \boxed{B = \frac{1}{e^{a}+e^{-a}}}$

$x = -e^{a}$: $\quad 1 = -A(e^{a}+e^{-a}) \quad \therefore \quad \boxed{A = -\frac{1}{e^{a}+e^{-a}}}$

$$\therefore \quad I = \frac{1}{e^{a}+e^{-a}} \int_1^{\infty} \left(\frac{-1}{x+e^{a}} + \frac{1}{x-e^{-a}} \right) dx$$

$$I = \frac{2}{\cosh a} \Big[\ln(x-e^{-a}) - \ln(x+e^{a}) \Big]_1^{\infty}$$

$$I = \frac{2}{\cosh a} \left[\ln\left(\frac{x-e^{-a}}{x+e^{a}} \right) \right]_1^{\infty}$$

$$I = \frac{2}{\cosh a} \left(\ln 1 - \ln\left(\frac{1-e^{-a}}{1+e^{a}} \right) \right)$$

$$I = -\frac{2}{\cosh a} \ln\left(\frac{e^{-a/2}(e^{a/2}-e^{-a/2})}{e^{a/2}(e^{-a/2}+e^{a/2})} \right)$$

$$I = -\frac{2}{\cosh a} \ln\left(e^{-a} \tanh \frac{a}{2} \right)$$

$$I = -\frac{2}{\cosh a} \left(-a + \ln\left(\tanh \frac{a}{2} \right) \right)$$

$$\therefore \boxed{I = \frac{2}{\cosh a}\left(a + \ln\left(\coth\frac{a}{2}\right)\right)}$$

$t \quad -\ln x = \ln x^{-1} \qquad \coth\frac{a}{2} = \frac{1}{\tanh\frac{a}{2}}]$

Now try $\quad I = \displaystyle\int_0^\infty \frac{1}{x^4+2x^2\cosh a+1}\,dx$

$$x^4 + 2x^2\cosh a + 1 = (x^2+e^a)(x^2+e^{-a})$$

as per (i)

$$\therefore \frac{1}{x^4+2x^2\cosh a+1} = \frac{A}{x^2+e^a} + \frac{B}{x^2+e^{-a}}$$

$$\therefore 1 = A(x^2+e^{-a}) + B(x^2+e^a)$$

let $x=0$: $\quad 1 = Ae^{-a} + Be^a \qquad (1)$

$x=1$: $\quad 1 = A(1+e^{-a}) + B(1+e^a) \qquad (2)$

(2) - (1): $\quad 0 = A + B \quad \therefore \boxed{B = -A}$

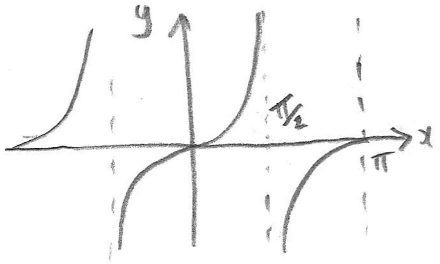
in (1): $\quad 1 = Ae^{-a} - Ae^a \quad \therefore A = \frac{-1}{e^a - e^{-a}}$

$$\therefore \boxed{A = \frac{-1}{2\sinh a}}$$

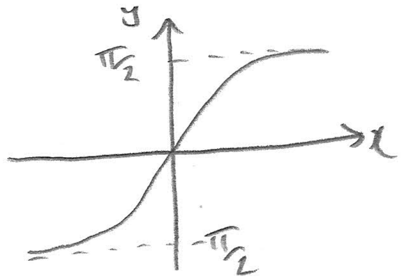
$$\text{So } I = \frac{-1}{2\sinh a}\int_0^\infty\left(\frac{1}{x^2+e^a} - \frac{1}{x^2+e^{-a}}\right)dx = \frac{-1}{2\sinh a}\int_0^\infty\left(\frac{1}{x^2+(e^{\frac{a}{2}})^2} - \frac{1}{x^2+(e^{-\frac{a}{2}})^2}\right)dx$$

Now $\boxed{\int\frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c}$ (Quote from formula book)

$$\therefore \quad I = \frac{-1}{2\sinh a}\left[e^{-\frac{a}{2}} \tan^{-1}\left(x e^{-a/2}\right) - e^{a/2} \tan^{-1}\left(x e^{a/2}\right)\right]_0^{\infty}$$



$y = \tan x$



$y = \tan^{-1} x$

So $\tan^{-1}(\infty) = \pi/2$ $\quad \left\{ \text{Strictly } \lim_{x \to \infty} (\tan^{-1} x) = \pi/2 \right\}$

$$\therefore \quad I = \frac{-1}{2\sinh a}\left(e^{-a/2}\frac{\pi}{2} - e^{+a/2}\frac{\pi}{2}\right)$$

$$I = \frac{1}{2\sinh a} \frac{\pi}{2}\left(e^{a/2} - e^{-a/2}\right)$$

$$I = \frac{\pi}{4\sinh a} \times 2\sinh a/2$$

Now $\quad \sinh a = 2\sinh a/2 \cosh a/2$

$$\therefore \quad \boxed{I = \frac{\pi}{4\cosh a/2}}$$