3/ n is a positive integer $(n \in \mathbb{Z}^+)$

a is a complex number $(a \in \mathbb{C})$

$a^n = 1$ s.t. $a^m = 1$; $0 < m < n$ and m is not an integer

"nth primitive root of unity" ← a

Now $i^4 = (i^2)^2 = (-1)^2 = 1$

$(-i)^4 = (i^2)^2 = 1$

So $\boxed{\pm i \text{ are } 4^{th} \text{ primitive roots of unity}}$

let $C_4 = (x-i)(x+i)$ ie a polynomial whose roots are the nth primitive roots of unity

$\Rightarrow \boxed{C_4(x) = x^2+1}$

Condition is the coefficient of highest powers of x is unity

ie C_4 could be $k(x-i)(x+i)$ but $k = 1$ by this condition.

{ let a_n be an nth primitive root of unity }

Now $a^1 = 1$ when $a = 1$

So $\boxed{C_1(x) = x - 1}$

lets generalize the idea and try and solve $x^n = 1$

ie $x^n - 1 = 0$ (or $1 - x^n = 0$)

To find the roots - and ∴ compare to our 'primitive' criterion we need to factorize $x^n - 1$

Consider the GP: $1 + x + x^2 + x^3 + \dots + x^{n-1} = \dfrac{1-x^n}{1-x}$

∴ $1 - x^n = (1-x)(1+x+x^2+x^3+\dots+x^{n-1})$

$1 - x = 1 - x$ Root $x=1$. $1^1 = 1$ so $\boxed{a_1 = 1}$

$1 - x^2 = (1-x)(1+x)$ Roots $x = 1, -1$. $(-1)^2 = 1$ so $\boxed{a_2 = -1}$

$1 - x^3 = (1-x)(1+x+x^2)$ Roots $x = 1, \frac{-1 \pm i\sqrt{3}}{2}$

$1 - x^4 = (1-x)(1+x+x^2+x^3)$

$1 - x^5 = (1-x)(1+x+x^2+x^3+x^4)$

$1 - x^6 = (1-x)(1 + x + x^2 + x^3 + x^4 + x^5)$

So

$$\boxed{\begin{aligned} C_1(x) &= x-1 \\ C_2(x) &= x+1 \\ C_3(x) &= 1+x+x^2 \\ C_5(x) &= 1+x+x^2+x^3+x^4 \end{aligned}}$$

Idea is we set $C_n(x)$ as the polynomial factor $1-x^n$ such that other factors are not already $C_m(x)$ where $m<n$

Now $1-x^4 = (1-x^2)(1+x^2)$

$= (1-x)(1+x)(1+x^2)$

$\therefore$ $\boxed{C_4(x) = 1+x^2}$ { consistent with part (i) }

$1 - x^6 = (1+x^3)(1-x^3)$

$= (1+x^3)(1-x)(1+x+x^2)$

But $(1+x^3) = (1+x)(1-x+x^2)$

So $1-x^6 = \underbrace{(x-1)}_{C_1(x)}\underbrace{(x+1)}_{C_2(x)}\underbrace{(1+x+x^2)}_{C_3(x)}(1-x+x^2)$

$\therefore$ $\boxed{C_6(x) = 1-x+x^2}$

(ii) let $C_n(x) = x^4+1$ when $C_n(x) = 0$

$\Rightarrow x^4 = -1$

$\therefore x^8 = 1$

So $\boxed{n=8}$

(iii) let p be a prime number

$x^p = 1 \Rightarrow x^p - 1 = 0$

$\Rightarrow x^p - 1 = (x-1)(1 + x + x^2 + \dots + x^{p-2} + x^{p-1})$

Consistent with the pattern above, if p is prime

$$\boxed{1 + x + x^2 + \dots x^{p-1} \doteq C_p(x)}$$

{ * I'm not sure how to rigorously prove this.....
but the idea is $1 + x + x^2 + \dots x^{p-1}$ cannot be factorised into $f(x)g(x)$ where $f(x)$ and $g(x)$ are polynomials with real coefficients? * }

(iv) Consider $C_q(x) = C_r(x)C_s(x)$ $r, s, q \in \mathbb{Z}^+$

This cannot be true given the {...} idea above.

ie $x^n - 1 = (x-1)(1 + x + x^2 + \dots x^{n-1})$

$= C_n(x) \times$ .....

Some product of $\{C_m(x)\}$, where $m < n$

ie not all $m < n$ are needed

If $q = r$ the $C_s(x)$ would have to equal 1

Since there are no s values where $C_s(x) = 1$ this is impossible.

(Same argument for $q = s$ ie $C_r(x) = 1$ which is impossible).