7/ $y = \cos(m \sin^{-1} x) \qquad |x| < 1$

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

(Quote from formula book
↳ or one's memory (!))

$$\therefore \frac{dy}{dx} = \boxed{\frac{-\sin(m \sin^{-1} x)\, m}{\sqrt{1-x^2}}}$$

$$\frac{d^2y}{dx^2} = \frac{\sqrt{1-x^2}\left(-\cos(m\sin^{-1}x)\frac{m^2}{\sqrt{1-x^2}}\right) + \sin(m\sin^{-1}x)\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)m}{1-x^2}$$

$$= \boxed{\frac{-\cos(m\sin^{-1}x)\,m^2}{1-x^2} - \frac{mx\sin(m\sin^{-1}x)}{(1-x)^{3/2}}}$$

$$\therefore (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2 y \qquad (*) \qquad \text{"}n=0\text{ case"}$$

$$= -\cos(m\sin^{-1}x)m^2 - \frac{mx\sin(m\sin^{-1}x)}{\sqrt{1-x}} - \frac{mx\sin(m\sin^{-1}x)}{\sqrt{1-x}}$$

$$+ m^2\cos(m\sin^{-1}x) = 0 \qquad \text{as required.}$$

Next part asks for similar relationships to (*) for $\frac{d^3y}{dx^3}$ and $\frac{d^4y}{dx^4}$. Since (*) is zero, simply differentiate it rather than using $y = \cos(m\sin^{-1}x)$. Life is short!!

ie $$(1-x^2)\frac{d^3y}{dx^3} - 2x\frac{d^2y}{dx^2} - x\frac{d^2y}{dx^2} - \frac{dy}{dx} + m^2\frac{dy}{dx} = 0$$

$$\boxed{(1-x^2)\frac{d^3y}{dx^3} - 3x\frac{d^2y}{dx^2} + (m^2-1)\frac{dy}{dx} = 0} \qquad \text{"}n=1\text{ case"}$$

$\therefore \quad (1-x^2)\dfrac{d^4y}{dx^2} - 2x\dfrac{d^3y}{dx^3} - 3x\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + (m^2-1)\dfrac{d^2y}{dx^2} = 0$

$$\Rightarrow \quad \boxed{(1-x^2)\frac{d^4y}{dx^4} - 5x\frac{d^3y}{dx^3} + (m^2-4)\frac{d^2y}{dx^2} = 0}$$

"$n=2$ case"

$\therefore$ conjecture (+)

$$\boxed{(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - (2n+1)x\frac{d^{n+1}y}{dx^{n+1}} + (m^2-n^2)\frac{d^ny}{dx^n} = 0}$$

clearly $n=0$ works (and $n=1$ and 2)

consider $n+1$ th case for (+)

(**)

is $\boxed{(1-x^2)\dfrac{d^{n+3}y}{dx^{n+3}} - (2n+3)x\dfrac{d^{n+2}}{dx^{n+2}} + (m^2-(n+1)^2)\dfrac{d^{n+1}y}{dx^{n+1}} = 0}$

Now differentiate (+)

$$(1-x^2)\frac{d^{n+3}y}{dx^{n+3}} - 2x\frac{d^{n+2}y}{dx^{n+2}} - (2n+1)x\frac{d^{n+2}y}{dx^{n+2}} - (2n+1)\frac{d^{n+1}y}{dx^{n+1}}$$

$$+ (m^2-n^2)\frac{d^{n+1}y}{dx^{n+1}} = 0$$

$$\Rightarrow \quad (1-x^2)\frac{d^{n+3}y}{dx^{n+1}} - (2n+3)x\frac{d^{n+2}y}{dx^{n+2}} + (m^2-n^2-2n-1)\frac{d^{n+1}y}{dx^{n+1}} = 0$$

$$\boxed{(1-x^2)\frac{d^{n+3}y}{dx^{n+1}} - (2n+3)x\frac{d^{n+2}y}{dx^{n+2}} + (m^2-(n+1)^2)\frac{d^{n+1}y}{dx^{n+1}} = 0}$$

is (**)

So conjecture holds for $n+1$ if n is true.

Now since $n=1$ is true $\therefore$ $n=2$ is true etc. $\therefore$ conjecture

(21) (by induction) is $\boxed{\text{true}}$ for all $n \in \mathbb{Z}^+$.

$$y = \cos(m\sin^{-1}x) = f(x) \qquad \boxed{f(0) = 1}$$

$$y = f(0) + xf'(0) + \frac{x^2 f''(0)}{2} + \frac{x^3 f^{(3)}(0)}{6} + \frac{x^4 f^{(4)}(0)}{24} + \ldots$$

(Maclaurin series)

$$f'(x) = \frac{-\sin(m\sin^{-1}x)m}{\sqrt{1-x^2}} \qquad \therefore \boxed{f'(0) = 0} \qquad [\sin^{-1}(0) = 0]$$

$$f''(x) = \frac{-\cos(m\sin^{-1}x)m^2}{1-x^2} - \frac{mx\sin(m\sin^{-1}x)}{(1-x)^{3/2}}$$

$$\therefore \boxed{f''(0) = -m^2} \qquad \left[f^{(n)}(x) \equiv \frac{d^n y}{dx^n}\right]$$

Now $(1-x^2)\frac{d^3y}{dx^3} - 3x\frac{d^2y}{dx^2} + (m^2-1)\frac{dy}{dx} = 0$

So when $x=0$: $f^{(3)}(0) - (m^2-1)(0) = 0 \quad \therefore \boxed{f^{(3)}(0) = 0}$

$$(1-x^2)f^{(4)}(x) - 5x f^{(3)}(x) + (m^2-4)f''(x) = 0$$

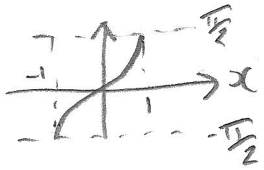
$\therefore$ when $x=0$: $f^{(4)}(0) + (m^2-4)(-m^2) = 0$

$$\therefore \boxed{f^{(4)}(0) = m^2(m^2-4)}$$

$$\therefore \boxed{y = 1 - \frac{m^2x^2}{2} + \frac{m^2(m^2-4)}{24}x^4 + \ldots}$$

Now $y = \cos(m\sin^{-1}x)$ so let $\theta = \sin^{-1}x$

$\therefore x = \sin\theta$.



$$\therefore \boxed{y = 1 - \frac{m^2\sin^2\theta}{2} + \frac{m^2(m^2-4)}{24}\sin^4\theta} + \ldots$$

as required

Now from the (proved) conjecture (†)

$$(1-x^2) f^{(n+2)}(x) - (2n+1)x f^{(n+1)}(x) + (m^2-n^2) f^{(n)}(x) = 0$$

$\therefore$ when $x = 0$

$$\boxed{f^{(n+2)}(0) = (n^2-m^2) f^{(n)}(0)}$$

$f(0) = 1 \Rightarrow$ only even differentials contribute.

$f'(1) = 0 \Rightarrow$ all odd n differentials are zero

let $n = 2k$, $k \in \mathbb{Z}^+$ to just count the even terms

$$\therefore f^{(2k+2)}(0) = (4k^2-m^2) f^{(2k)}(0)$$

$$\therefore f^{(2k+2)}(0) = -m^2(4-m^2)(16-m^2)(32-m^2)\cdots(4k^2-m^2)$$

But when $n = m$ (or $2k = m$), $n^2-m^2 = 0$

So sequence will terminate. ie $f^{(p)}(0) = 0$ for $p \geq m$.

So last term of the sequence, if m is even, is ..... $f^{(m)}(0)$, since $f^{(m+2)}(0) = 0$, and $\underbrace{f^{(m+1)}(0) = 0}_{\text{always}}$ if m is even.

$$\boxed{\text{So } \cos m\theta \text{ is a finite polynomial of order } m \text{ in } \sin\theta \text{ if } m \text{ is even.}}$$