12/ $S = 1 + (1+d)r + (1+2d)r^2 + \dots + (1+nd)r^n + \dots$

s.t. $|r| < 1$

$rS = r + (1+d)r^2 + \dots + (1+(n-1)d)r^n + \dots$

$\therefore \; S - rS = 1 + d(r + r^2 + r^3 + \dots + r^n + \dots)$

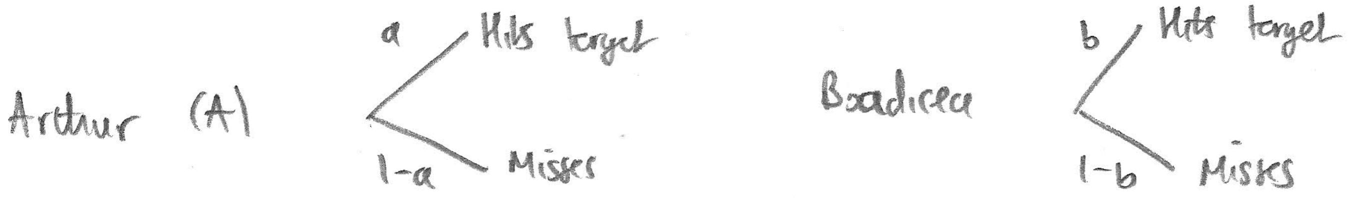
Now $(\dots)$ is an infinite GP -1

ie $S - rS = 1 + d\left(\frac{1}{1-r} - 1\right)$

Since if $|r|$, $1 + r + r^2 + \dots r^n + \dots = \frac{1}{1-r}$

$\therefore \; S(1-r) = 1 + \frac{d(1-1+r)}{1-r}$

$\boxed{S = \frac{1}{1-r} + \frac{rd}{(1-r)^2}}$ as required.



All shots are independent.

let x be the number of shots it takes Arthur to hit the target.

$\therefore \; P(x) = (1-a)^{x-1} a$ ie $x \sim Geo(a)$

$E[x] = \sum_{x=1}^{\infty} x\,P(x) = a \sum_{x=1}^{\infty} x(1-a)^{x-1}$

$\therefore \; E[x] = a + 2(1-a)a + 3(1-a)^2 a + 4(1-a)^3 a + \dots$

So $\frac{E[x]}{a} = 1 + 2(1-a) + 3(1-a)^2 + 4(1-a)^3 + \dots$

Now compare with initial result

$$\frac{1}{1-r} + \frac{rd}{(1-r)^2} = 1 + (1+d)r + (1+2d)r^2 + (1+3d)r^3 + \dots$$

$1+d = 2 \Rightarrow d=1$

$1+2d = 3 \Rightarrow d=1$

$1+3d = 4 \Rightarrow d=1$

So let $d=1$, $r = 1-a$

$$\therefore \quad E[x] = a\left\{\frac{1}{1-(1-a)} + \frac{(1-a)}{(1-(1-a))^2}\right\}$$

$$= a\left\{\frac{1}{a} + \frac{1-a}{a^2}\right\}$$

$$= 1 + \frac{1-a}{a}$$

$$= \frac{a+1-a}{a}$$

$$= \boxed{\frac{1}{a}}$$ as required.

A and B have a contest, A goes first. They take alternate shots. $P(A \text{ wins}) = \alpha$ and $P(B \text{ wins}) = \beta$. To win, you hit the target first. Clearly $\boxed{\beta = 1-\alpha}$

$$\alpha = \underbrace{a}_{\text{Win on first go}} + \underbrace{(1-a)(1-b)a}_{\text{Neither player wins, then A wins}} + \underbrace{(1-a)^2(1-b)^2 a}_{\text{Neither player wins first two rounds, A wins}} + (1-a)^3(1-b)^3 a + $$

This is a GP

So $\boxed{\alpha = \dfrac{a}{1-(1-a)(1-b)}}$ as required

$$\therefore \quad \beta = 1 - \alpha = 1 - \frac{a}{1-(1-a)(1-b)}$$

$$= \frac{1-(1-a)(1-b) - a}{1-(1-a)(1-b)}$$

$$= \frac{1-a - (1-a)(1-b)}{1-(1-a)(1-b)}$$

$$\boxed{\beta = \frac{(1-a)b}{1-(1-a)(1-b)}}$$

Now the expected number of shots is $E[S]$

$$E[S] = \underset{\text{A wins}}{a} + \underset{\substack{\text{A misses}\\ \text{B wins}}}{2(1-a)b} + \underset{\substack{\text{A misses}\\ \text{B misses}\\ \text{A wins}}}{3(1-a)(1-b)a} + 4(1-a)(1-b)(1-a)b$$

$$+ 5(1-a)^2(1-b)^2 a$$

$$+ 6(1-a)^3(1-b)^2 b$$

$$+ \cdots$$

$$E[S] = a\left\{1 + 3(1-a)(1-b) + 5(1-a)^2(1-b)^2 + 7(1-a)^3(1-b)^3 + \cdots\right\}$$

$$+ 2b(1-a)\left\{1 + 2(1-a)(1-b) + 3(1-a)^2(1-b)^2 + \cdots\right\}$$

using the original result $\quad a\{\ldots\} \quad r = (1-a)(1-b) \quad d = 2$

(40) $\quad b\{\ldots\} \quad r = (1-a)(1-b) \quad d = 1$

$$\therefore \quad E[S] = a\left\{ \frac{1}{1-(1-a)(1-b)} + \frac{2(1-a)(1-b)}{\left(1-(1-a)(1-b)\right)^2} \right\}$$

$$+ 2b\left\{ \frac{1}{1-(1-a)(1-b)} + \frac{(1-a)(1-b)}{\left(1-(1-a)(1-b)\right)^2} \right\}(1-a)$$

We must show this equals $\frac{\alpha}{a} + \beta/b$

$$\alpha = \frac{a}{1-(1-a)(1-b)} \qquad \beta = \frac{(1-a)b}{1-(1-a)(1-b)} \qquad \therefore \beta/\alpha = \frac{b(1-a)}{a}$$

$$E[S] = \frac{a + 2b(1-a)}{1-(1-a)(1-b)} + \frac{2a(1-a)(1-b) + 2b(1-a)^2(1-b)}{\left(1-(1-a)(1-b)\right)^2}$$

Now $\frac{\alpha}{a} + \beta/b = \frac{\alpha}{a} + \frac{\alpha(1-a)}{a} = \frac{\alpha(2-a)}{a}$

$$\therefore \boxed{\frac{\alpha}{a} + \frac{\beta}{b} = \frac{2-a}{1-(1-a)(1-b)}}$$

$$E[S] = \frac{1}{1-(1-a)(1-b)}\left\{ a + 2b(1-a) + \frac{2a(1-a)(1-b) + 2b(1-a)^2(1-b)}{1-(1-a)(1-b)} \right\}$$

$$= \frac{1}{1-(1-a)(1-b)} \cdot \frac{\left\{ a - a(1-a)(1-b) + 2b(1-a) - \cancel{2b(1-a)^2(1-b)} + 2a(1-a)(1-b) + \cancel{2b(1-a)^2(1-b)} \right\}}{1-(1-a)(1-b)}$$

$$= \frac{1}{\left(1-(1-a)(1-b)\right)^2}\left(a + 2b(1-a) + a(1-a)(1-b) \right)$$

$$(2-a)(1-(1-a)(1-b)) = 2-a-2(1-a)(1-b)+a(1-a)(1-b)$$

$$= 2-a-2(1-a)+2b(1-a)+a(1-a)(1-b)$$

$$= 2-a-2+2a+2b(1-a)+a(1-a)(1-b)$$

$$= a+2b(1-a)+a(1-a)(1-b)$$

$$\therefore \quad E[S] = \frac{2-a}{1-(1-a)(1-b)} = \boxed{\frac{\alpha}{a}+\frac{\beta}{b}}$$

as required