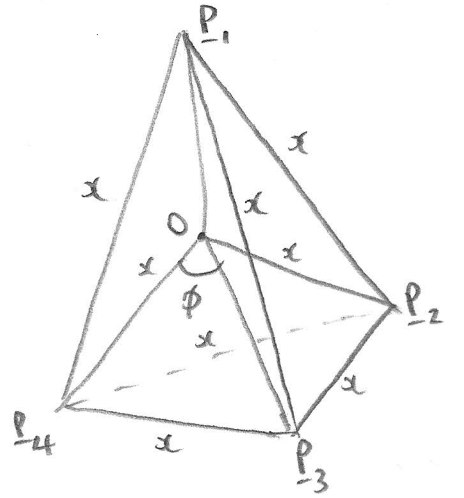
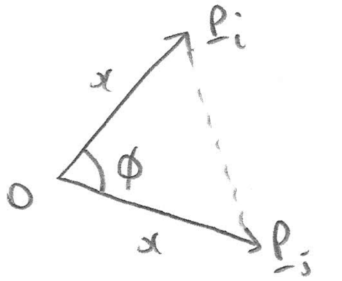
3/



O is the centre of a regular tetrahedron.



Now $\underline{P}_i \cdot \underline{P}_j = x^2 \cos\phi \qquad i \neq j$

$\underline{P}_i \cdot \underline{P}_j = x^2 \qquad i = j$

Since O is the centre of the tetrahedron $\underline{P}_1 + \underline{P}_2 + \underline{P}_3 + \underline{P}_4 = \underline{0}$

$\therefore \sum_{i=1}^{4} \underline{P}_i = \underline{0} \qquad \therefore \sum_{i=1}^{4} \underline{P}_i \cdot \underline{P}_j = 0$

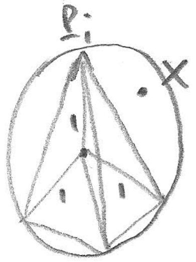
$\Rightarrow 3x^2 \cos\phi + x^2 = 0$

$\Rightarrow x^2 (3\cos\phi + 1) = 0$

$\therefore$ Since $x > 0 \Rightarrow \boxed{\cos\phi = -\frac{1}{3}}$

Now if $\{\underline{P}_i\}$ are on the surface of a sphere of unit radius, with O at the centre, $x = 1$

$\therefore \boxed{\underline{P}_i \cdot \underline{P}_j = -\frac{1}{3}} \quad i \neq j$ as required.



(i) let X be any point on the unit sphere surface. let $\underline{x}$ be the position vector of X from O

$|XP_i|^2 = (\underline{P}_i - \underline{x}) \cdot (\underline{P}_i - \underline{x}) = |\underline{P}_i|^2 - 2\underline{x} \cdot \underline{P}_i + |\underline{x}|^2$

$= 1 - 2\underline{x} \cdot \underline{P}_i + 1$

$= 2(1 - \underline{x} \cdot \underline{P}_i)$

$$\therefore \quad \sum_{i=1}^{4} |\underline{x} + \underline{p}_i|^2 = 8 - 2\underline{x} \cdot \sum_{i=1}^{4} \underline{p}_i$$

$$= \boxed{8} \qquad \text{Since } \sum_{i=1}^{4} \underline{p}_i = \underline{0}$$

(iii) let $\underline{p}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\underline{p}_2 = \begin{pmatrix} a \\ 0 \\ b \end{pmatrix}$; $a > 0$

$|\underline{p}_2|^2 = 1 \quad \therefore \quad a^2 + b^2 = 1 \quad \Rightarrow \quad a = \sqrt{1 - b^2}$

$\underline{p}_1 \cdot \underline{p}_2 = -\frac{1}{3} \quad \therefore \quad \boxed{b = -\frac{1}{3}}$

$\therefore \quad a = \sqrt{1 - \frac{1}{9}} = \frac{\sqrt{8}}{3} = \boxed{\frac{2\sqrt{2}}{3}}$

let $\underline{p}_3 = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ $\qquad |\underline{p}_3|^2 = 1 \Rightarrow \boxed{\alpha^2 + \beta^2 + \gamma^2 = 1}$

$\underline{p}_3 \cdot \underline{p}_1 = -\frac{1}{3} \quad \therefore \quad \boxed{\gamma = -\frac{1}{3}}$

$\underline{p}_3 \cdot \underline{p}_2 = -\frac{1}{3} \quad \therefore \quad \frac{2\alpha\sqrt{2}}{3} + \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right) = -\frac{1}{3}$

$$\frac{2\alpha\sqrt{2}}{3} = -\frac{1}{3} - \frac{1}{9}$$

$$2\alpha\sqrt{2} = -1 - \frac{1}{3}$$

$$2\alpha\sqrt{2} = -\frac{4}{3}$$

$$\alpha = -\frac{2}{3\sqrt{2}}$$

$$\boxed{\alpha = -\frac{\sqrt{2}}{3}}$$

$\therefore \quad \beta = \pm\sqrt{1-\gamma^2-\alpha^2}$

$\beta = \pm\sqrt{1-\frac{1}{9}-\frac{2}{9}}$

$\beta = \pm\sqrt{\frac{6}{9}} = \pm\sqrt{\frac{3\times 2}{3\times 3}}$

$\boxed{\beta = \pm\frac{\sqrt{2}}{\sqrt{3}}}$

take + root for $\underline{p}_3$

$\therefore$ − root for $\underline{p}_4$ (as one would use the same method to find it)

So

$$\boxed{\underline{p}_3 = \begin{pmatrix} -\sqrt{2}/3 \\ \sqrt{2}/\sqrt{3} \\ -\frac{1}{3} \end{pmatrix} \qquad \underline{p}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \underline{p}_4 = \begin{pmatrix} -\sqrt{2}/3 \\ -\sqrt{2}/\sqrt{3} \\ -\frac{1}{3} \end{pmatrix} \qquad \underline{p}_2 = \begin{pmatrix} 2\sqrt{2}/3 \\ 0 \\ -\frac{1}{3} \end{pmatrix}}$$

(iii) $\displaystyle\sum_{i=1}^{4} XP_i^{\,4} = \sum_{i=1}^{4}\left[(\underline{x}-\underline{p}_i)\cdot(\underline{x}-\underline{p}_i)\right]^2$

$\displaystyle = \sum_{i=1}^{4}\left[|\underline{x}|^2 - 2\underline{x}\cdot\underline{p}_i + |\underline{p}_i|^2\right]^2$

$\displaystyle = \sum_{i=1}^{4} 2^2\left(1-\underline{x}\cdot\underline{p}_i\right)^2$

$\displaystyle = \boxed{4\sum_{i=1}^{4}\left(1-\underline{x}\cdot\underline{p}_i\right)^2}$ as required

let $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\underline{p}_i$ be the coordinates above

$1-\underline{x}\cdot\underline{P}_1 = 1-z$

$1-\underline{x}\cdot\underline{P}_2 = 1-\frac{2\sqrt{2}}{3}x+\frac{1}{3}z$

$1-\underline{x}\cdot\underline{P}_3 = 1+\frac{\sqrt{2}}{3}x-\frac{\sqrt{2}}{\sqrt{3}}y+\frac{1}{3}z$

$1-\underline{x}\cdot\underline{P}_4 = 1+\frac{\sqrt{2}}{3}x+\frac{\sqrt{2}}{\sqrt{3}}y+\frac{1}{3}z$

$(1-z)^2 = \boxed{1-2z+z^2}$

	1	$-\frac{2\sqrt{2}}{3}x$	$\frac{1}{3}z$
1	1	$-\frac{2\sqrt{2}}{3}x$	$\frac{1}{3}z$
$-\frac{2\sqrt{2}}{3}x$	$-\frac{2\sqrt{2}}{3}x$	$\frac{8}{9}x^2$	$\frac{-2\sqrt{2}xz}{9}$
$\frac{1}{3}z$	$\frac{z}{3}$	$\frac{-2\sqrt{2}xz}{9}$	$\frac{z^2}{9}$

So $\left(1-\frac{2\sqrt{2}}{3}x+\frac{1}{3}z\right)^2$

$$= \boxed{1-\frac{4\sqrt{2}}{3}x+\frac{8}{9}x^2+\frac{2z}{3}-\frac{4\sqrt{2}xz}{9}+\frac{z^2}{9}}$$

	1	$\frac{\sqrt{2}}{3}x$	$\pm\frac{\sqrt{2}}{\sqrt{3}}y$	$\frac{1}{3}z$
1	1	$\frac{\sqrt{2}}{3}x$	$\pm\frac{\sqrt{2}}{\sqrt{3}}y$	$\frac{1}{3}z$
$\frac{\sqrt{2}}{3}x$	$\frac{\sqrt{2}}{3}x$	$\frac{2x^2}{9}$	$\pm\frac{2xy}{3\sqrt{3}}$	$\frac{\sqrt{2}}{9}xz$
$\pm\frac{\sqrt{2}}{\sqrt{3}}y$	$\pm\frac{\sqrt{2}}{\sqrt{3}}y$	$\pm\frac{2xy}{3\sqrt{3}}$	$\frac{2}{3}y^2$	$\pm\frac{\sqrt{2}}{3\sqrt{3}}yz$
$\frac{1}{3}z$	$\frac{1}{3}z$	$\frac{\sqrt{2}}{9}xz$	$\pm\frac{\sqrt{2}}{3\sqrt{3}}zy$	$\frac{1}{9}z^2$

All $\pm$ terms will cancel when we sum $(1-\underline{x}\cdot\underline{P}_3)^2 + (1-\underline{x}\cdot\underline{P}_4)^2$

So $\displaystyle\sum_{i=1}^{4}(1-\underline{x}\cdot\underline{P}_i)^2 = 1-2z+z^2+1-\frac{4\sqrt{2}x}{3}+\frac{8}{9}x^2+\frac{2z}{3}-\frac{4\sqrt{2}xz}{9}+\frac{z^2}{9}$

$\displaystyle+\left(1+\frac{2\sqrt{2}}{3}x+\frac{2x^2}{9}+\frac{2z}{3}+\frac{2}{3}y^2+\frac{1}{9}z^2+\frac{2\sqrt{2}xz}{9}\right)+2$

$$= 4 + \cancel{x\left(-\frac{4\sqrt{2}}{3} + \frac{4\sqrt{2}}{3}\right)} + \cancel{y(0)} + \cancel{z\left(-2 + \frac{2}{3} + \frac{4}{3}\right)}$$

$$+ \cancel{xz\left(-\frac{4\sqrt{2}}{9} + \frac{4\sqrt{2}}{9}\right)} + x^2\left(\frac{8}{9} + \frac{4}{9}\right) + y^2\left(\frac{4}{3}\right) + z^2\left(1 + \frac{1}{9} + \frac{2}{9}\right)$$

$$= 4 + \frac{4}{3}x^2 + \frac{4}{3}y^2 + \frac{4}{3}z^2 = 4 + \frac{4}{3}(x^2+y^2+z^2) = \boxed{4\frac{4}{3}}$$

since $|\underline{x}|^2 = 1$

$$\therefore \sum_{i=1}^{4} XP_i^4 = 4\sum_{i=1}^{4}(1 - \underline{x}\cdot\underline{p}_i)^2$$

$$= 4 \times \frac{16}{3}$$

$$= \boxed{\frac{64}{3}}$$

which is independent of x, y, z and $\therefore$ the position of X.