7/ (i) $\boxed{\frac{d^2y}{dx^2} + y^3 = 0}$ $\quad x = 0,\ y = 1$, $\left.\frac{dy}{dx}\right|_{x=0} = 0$ } "Boundary" conditions

Define $\boxed{E(x) = \left(\frac{dy}{dx}\right)^2 + \frac{1}{2}y^4}$

$$\frac{dE}{dx} = 2\frac{dy}{dx}\frac{d^2y}{dx^2} + 2y^3\frac{dy}{dx}$$

$$= 2\frac{dy}{dx}\underbrace{\left(\frac{d^2y}{dx^2} + y^3\right)}_{\text{zero}}$$

$\therefore \frac{dE}{dx} = 0 \Rightarrow \boxed{E = \text{Constant}}$

Now using "Boundary" conditions

$E(0) = 0 + \frac{1}{2}(1)^4$ so $\boxed{E = \frac{1}{2}}$

Now $\therefore \frac{1}{2} = \left(\frac{dy}{dx}\right)^2 + \frac{1}{2}y^4$

$\frac{1}{2}(1 - y^4) = \left(\frac{dy}{dx}\right)^2$

Now $\left(\frac{dy}{dx}\right)^2 \geq 0$

$\therefore 1 - y^4 \geq 0$

$1 \geq y^4$

$\boxed{1 \geq |y|}$ as required

(ii) $\boxed{\frac{d^2v}{dx^2} + x\frac{dv}{dx} + \sinh v = 0}$

S.t $v = \ln 3,\ \frac{dv}{dx} = 0,\ x = 0$

$\boxed{E(x) = \left(\frac{dv}{dx}\right)^2 + 2\cosh v}$

$$\frac{dE}{dx} = 2\frac{dv}{dx}\frac{d^2v}{dx^2} + 2\sinh v\frac{dv}{dx}$$

$$= 2\frac{dv}{dx}\left(\frac{d^2v}{dx^2} + \sinh v\right)$$

From original equation

$\frac{d^2v}{dx^2} + \sinh v = -x\frac{dv}{dx}$

so $\frac{dE}{dx} = 2\frac{dv}{dx}\left(-x\frac{dv}{dx}\right)$

$\boxed{\frac{dE}{dx} = -2x\left(\frac{dv}{dx}\right)^2}$

Since $\left(\frac{dv}{dx}\right)^2 \geq 0$ $\quad \boxed{\frac{dE}{dx} \leq 0 \text{ if } x \geq 0}$ as required

So Since E is decreasing for $x \geq 0$ then maximum value (in this range) is for $x=0$

$\therefore$ if $x \geq 0$,

$$E \leq E(0)$$

$$E \leq 2\cosh \ln 3$$

$$E \leq e^{\ln 3} + e^{-\ln 3}$$

$$E \leq 3 + \frac{1}{3}$$

$$E \leq \frac{10}{3}$$

$$\therefore \left(\frac{dv}{dx}\right)^2 + 2\cosh v \leq \frac{10}{3} \qquad (x \geq 0)$$

$$\left(\frac{dv}{dx}\right)^2 \leq \frac{10}{3} - 2\cosh v$$

Now Since $\left(\frac{dv}{dx}\right)^2 \geq 0$

$$\therefore \frac{10}{3} - 2\cosh v \geq 0$$

$$\therefore \boxed{\cosh v \leq \frac{5}{3} \quad \text{for } x \geq 0}$$ as required.

(iii) $$\boxed{\frac{d^2w}{dx^2} + (5\cosh x - 4\sinh x - 3)\frac{dw}{dx} + (w\cosh w + 2\sinh w) = 0} \quad (*)$$

S.t. $\frac{dw}{dx} = \frac{1}{\sqrt{2}}$, $w = 0$, $x = 0$

In previous parts, if $\frac{d^2y}{dx^2} + g(x)\frac{dy}{dx} + f(y) = 0$

$E(x) = 2\int f(y)\,dy + \left(\frac{dy}{dx}\right)^2$ let's try this approach here.

ii) $E(x) = 2\int (w\cosh w + 2\sinh w)\,dw + \left(\frac{dw}{dx}\right)^2$ (ignore constant of integration)

$$= 2\left(w\sinh w - \int \sinh w\,dw\right) + 4\cosh w + \left(\frac{dw}{dx}\right)^2$$

$$= 2w\sinh w - 2\cosh w + 4\cosh w + \left(\frac{dw}{dx}\right)^2$$

$$= \boxed{2(w\sinh w + \cosh w) + \left(\frac{dw}{dx}\right)^2}$$

$$\therefore \frac{dE}{dx} = 2\left(w\cosh w\frac{dw}{dx} + \sinh w\frac{dw}{dx} + \sinh w\frac{dw}{dx}\right) + 2\frac{dw}{dx}\frac{d^2w}{dx^2}$$

$$= 2\frac{dw}{dx}\left(\frac{d^2w}{dx^2} + w\cosh w + 2\sinh w\right)$$

using (*)

$$\frac{dE}{dx} = 2\left(\frac{dw}{dx}\right)^2\left(4\sinh x + 3 - 5\cosh x\right)$$

To find out if this increases or decreases for $x \geq 0$ we need to expand the $(\ldots)$

$$4\sinh x + 3 - 5\cosh x = 2e^x - 2e^{-x} + 3 - \frac{5}{2}e^x - \frac{5}{2}e^{-x}$$

$$= \frac{e^{-x}}{2}\left(4e^{2x} - 4 + 6e^x - 5e^{2x} - 5\right)$$

$$= \frac{e^{-x}}{2}\left(-e^{2x} + 6e^x - 9\right)$$

$$= -\frac{e^{-x}}{2}\left(e^{2x} - 6e^x + 9\right)$$

$$= -\frac{e^{-x}}{2}\left(e^x - 3\right)^2$$

Since $e^{-x} \geq 0$ and $(e^x - 3)^2 \geq 0 \quad \forall x$

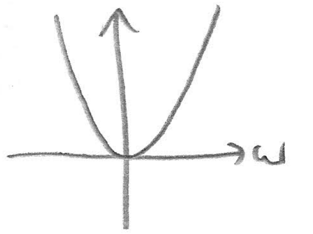
$$\Rightarrow \boxed{\frac{dE}{dx} \leq 0 \quad \forall x}$$

So for $x \geq 0$, $E(0)$ will be the highest value

$\therefore$ $2(w\sinh w + \cosh w) + \left(\frac{dw}{dx}\right)^2 \leq E(0)$

$E(0) = 2 + \frac{1}{2} = \frac{5}{2}$

$\therefore$ $\left(\frac{dw}{dx}\right)^2 \leq \frac{5}{2} - 2w\sinh w - 2\cosh w$



Now $w\sinh w$ is even and ≥ 0

So $\frac{5}{2} - 2\cosh w \geq 0$

$\therefore$ $\boxed{\cosh w \geq \frac{5}{4}}$ for $x \geq 0$

as required.