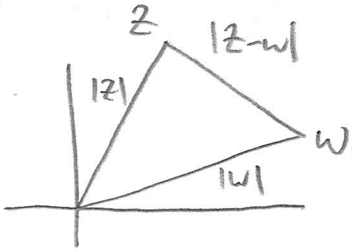
6/ The result given $|z-w| \leq |z| + |w|$ (z, w are complex numbers) follows from the more familiar TRIANGLE INEQUALITY

$$|z+w| \leq |z| + |w|$$

but when $w \to -w$

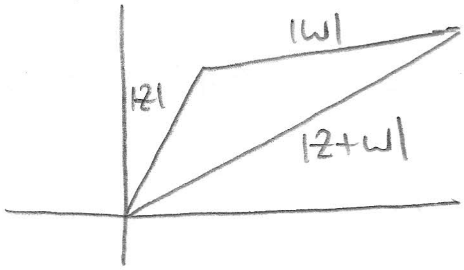
ie $$|z-w| \leq |z| + |-w|$$

$$|z-w| \leq |z| + |w|$$



This is fairly obvious geometrically

as is



[Proof using algebra: $|z+w|^2 = (z+w)(z^*+w^*)$

↑ $= zz^* + (wz^* + zw^*) + ww^*$

This will be useful later] $= |z|^2 + |w|^2 + wz^* + zw^*$

let $z = |z|e^{i\theta}$
$w = |w|e^{i\phi}$

$\therefore wz^* + zw^* = |z||w|(e^{i(\theta-\phi)} + e^{-i(\theta-\phi)})$

$= 2|z||w| \cos(\theta - \phi)$

Now $Re(wz^*) = Re(zw^*) = |z||w|\cos(\theta-\phi)$

So $$\boxed{|z+w|^2 = |z|^2 + |w|^2 + 2Re(zw^*)}$$

Now max value of $\cos(\theta-\phi)$ is 1, so $Re(zw^*) \leq |z||w|$

$\therefore$ Since $|z+w| - |z|^2 - |w|^2 = 2Re(zw^*)$

$$|z+w| - |z|^2 - |w|^2 \leq 2|z||w|$$

$$\therefore |z+w|^2 \leq |z|^2 + |w|^2 + 2|z||w|$$

$\therefore \quad |z+w|^2 \le (|z|+|w|)^2$

$\therefore \quad \boxed{|z+w| \le |z|+|w|}$ as required.

Note $\quad |z-w|^2 = (z-w)(z^*-w^*)$

$= |z|^2+|w|^2-(wz^*+zw^*)$

$= |z|^2+|w|^2-2\,\mathrm{Re}(zw^*)$

$\therefore$ Since $\quad 2\,\mathrm{Re}(zw^*) = |z|^2+|w|^2-|z-w|^2$

and $\quad \mathrm{Re}(zw^*) \le |z||w|$

$\therefore \quad |z|^2+|w|^2-|z-w|^2 \le |z||w|$

$\therefore \quad |z-w|^2 \ge |z|^2+|w|^2-|z||w|$

$\ge (|z|-|w|)^2$

$\therefore \quad \boxed{|z-w| \ge |z|-|w|}$ as well

So $\quad \boxed{|z|-|w| \le |z-w| \le |z|+|w|}$]

(i) $\quad |z-w|^2 = (z-w)(z^*-w^*)$

$= |z|^2+|w|^2-(wz^*+zw^*)$

Now let $\quad E = zw^*+z^*w+2|zw|$

$\therefore \quad zw^*+z^*w = E-2|zw|$

$\therefore \quad |z-w|^2 = |z|^2+|w|^2-E+2|zw|$

Now if $z = |z| e^{i\theta}$

$w = |w| e^{i\phi}$

$zw = |z||w| e^{i(\theta+\phi)}$

$\therefore \; |zw| = |z||w| \, |e^{i(\theta+\phi)}|$

$\therefore \; \boxed{|zw| = |z||w|}$

$\therefore \; |z-w|^2 = |z|^2 + |w|^2 + 2|z||w| - E$

$\boxed{|z-w|^2 = (|z| + |w|)^2 - E}$ as required

Now from above: $wz^* + zw^* = 2\,\mathrm{Re}(zw^*)$

$\therefore \; E = 2\,\mathrm{Re}(zw^*) + 2|z||w|$ So $\boxed{E \text{ is real}}$

Now since $\mathrm{Re}(zw^*) = |z||w| \cos(\theta - \phi)$

$-|z||w| \leq \mathrm{Re}(zw^*) \leq |z||w|$

Since $-1 \leq \cos(\theta - \phi) \leq 1$

$\therefore \; \boxed{E \geq 0}$ So $\boxed{E \text{ is real and positive}}$

(ii) $|1 - zw^*|^2 = (1 - zw^*)(1 - z^*w)$

$= 1 - zw^* - z^*w + |z||w|$

$= 1 - (E - 2|z||w|) + |z|^2|w|^2$

$= 1 + 2|z||w| + (|z||w|)^2 - E$

$= \boxed{(1 + |zw|)^2 - E}$

Again use $|z||w| = |zw|$

So $|z-w|^2 = (|z|+|w|)^2 - E$ (i)

$|1-zw^*|^2 = (1+|zw|)^2 - E$ (ii)

Conjecture $\boxed{\dfrac{|z-w|}{|1-zw^*|} \leq \dfrac{|z|+|w|}{1+|zw|}}$

Since both sides are positive

$$|z-w|^2(1+|zw|)^2 \leq |1-zw^*|^2(|z|+|w|)^2$$

$$\therefore \left((|z|+|w|)^2 - E\right)\left(1+2|z||w|+|zw|^2\right)$$

$$\leq \left((1+|zw|)^2 - E\right)\left(|z|^2+2|zw|+|w|^2\right)$$

$$\Rightarrow E\left(1+\cancel{2|z||w|}+|zw|^2 - |z|^2 - \cancel{2|zw|} - |w|^2\right)$$

$$\geq (|z|+|w|)^2(1+|zw|)^2 - (1+|zw|)^2(|z|+|w|)^2$$

better to re-factorize

$$\Rightarrow E\left(1-|w|^2\right)\left(1-|z|^2\right) \geq 0$$

Now E is real and positive, so conjecture is true if $\boxed{(1-|w|^2)(1-|z|^2) \geq 0}$ (*)

if $|z|>1$ and $|w|>1$ then both $(\dots)$ are <0

so (*) is true.

if $|z|<1$ and $|w|<1$ then both $(\dots)$ are >0

so (*) is also true.