13/ a) X is a continuous random variable $\boxed{0 \le X \le 1}$

$$P(x \le X) = \int_0^X f(x)\,dx$$

$f(x)$ is probability density. We are given $\boxed{0 \le f(x) \le M}$

$$\boxed{F(x) = \int_0^x f(t)\,dt}$$

Cumulative distribution function.

(i)

$$0 \le f(t) \le M$$

$$0 \le \int_0^x f(t)\,dt \le M \int_0^x dt$$

$$\boxed{0 \le F(x) \le Mx}$$

as required.

(ii) $$I = \int_0^1 2g(x)\,F(x)\,f(x)\,dx = \Big[\, 2g(x)\,F(x) \times \underset{\int f(x)dx}{\underset{\uparrow}{F(x)}} - \int \frac{d}{dx}[2gF]\,F\,dx \Big]_0^1$$

Now $$\int f(x)\,dx = F(x)$$

$$\frac{d}{dx}F(x) = f(x)$$

$$\therefore I = \Big[\, 2g(x)\,F^2(x) - \int (2gf + 2Fg')\,F\,dx \Big]_0^1$$

$$I = \big[2gF^2\big]_0^1 - \underbrace{\int_0^1 2gfF\,dx}_{I} - 2\int_0^1 F^2 g'\,dx$$

$$2I = 2g(1)F(1)^2 - 2g(0)F(0)^2 - 2\int_0^1 g'(x)\,(F(x))^2\,dx$$

Now $F(0) = 0$

$F(1) = 1$

$$\therefore \boxed{I = g(1) - \int_0^1 g'(x)\,(F(x))^2\,dx}$$

as required.

b) Continuous random variable Y satisfies $0 \leq Y \leq 1$

$$P(y \leq Y) = \int_0^Y \underbrace{kF(y)\, f(y)}_{\text{probability density function.}}\, dy$$

$F(y)$ and $f(y)$ are as above.

(i) So $\boxed{\int_0^1 kF(y)\, f(y)\, dy = 1}$ since kFf is the PDF.

Now from part a) (ii), using $2g(y) = k$ $\therefore g'(y) = 0$

$$\int_0^1 kF(y)\, f(y)\, dy = g(1) - \underbrace{\int_0^1 g'(y)\, (F(y))^2\, dy}_{\text{zero since } g'(y)=0}$$

So since $g = k/2$

$\Rightarrow$ $g(1) = k/2$

So $\int_0^1 k\, F(y)\, f(y) dy = \frac{k}{2}$ $\therefore$ $\frac{k}{2} = 1 \Rightarrow \boxed{k = 2}$

(ii) $E[Y^n] = \int_0^1 2y^n\, F(y)\, f(y)\, dy$

Now in a) (i) $F(x) \leq Mx$ so $F(y) \leq My$

$\therefore$ $E[Y^n] \leq \int_0^1 2y^{n+1} M\, f(y)\, dy$

$E[Y^n] \leq 2M \int_0^1 y^{n+1} f(y)\, dy$

$\boxed{E[Y^n] \leq 2M\, \mu_{n+1}}$ where $\mu_n = E[x^n] = \int_0^1 x^n f(x)\, dx$.

$$E(Y^n) = \underbrace{1}_{g(y)=y^n \therefore g(1)=1} - \int_0^1 n y^{n-1} F^2(y)\,dy$$

[using result in (a) (ii)]

$\hookrightarrow g(y) = y^n$

Now again since $F(y) \le My$

$$\int_0^1 n y^{n-1} F^2 dy \le \int_0^1 n y^{n-1} My\, F(y)\,dy$$

$$\le nM \int_0^1 y^n F(y)\,dy$$

$$\le nM \left(\left[\frac{1}{n+1} y^{n+1} F(y) \right]_0^1 - \int_0 \frac{y^{n+1}}{n+1} f(y)\,dy \right)$$

$$\frac{dF}{dy} = f(y)$$

$$\le nM \left(\frac{1}{n+1} - \frac{1}{n+1} \mu_{n+1} \right)$$

So $\int_0^1 n y^{n-1} F^2 dy = 1 - E[Y^n]$

$$\therefore 1 - E[Y^n] \le \frac{nM}{n+1} - \frac{nM}{n+1}\mu_{n+1}$$

$$\therefore \boxed{E[Y^n] \ge 1 + \frac{nM\mu_{n+1}}{n+1} - \frac{nM}{n+1}}$$

Hence:

$$\boxed{1 + \frac{nM}{n+1}\mu_{n+1} - \frac{nM}{n+1} \le E[Y^n] \le 2M\mu_{n+1}}$$

as required

(iii) So if $n \geqslant 1$ the inequality above means

$$2M\mu_{n+1} \geqslant 1 + \frac{nM}{n+1}\mu_{n+1} - \frac{nM}{n+1}$$

let $n+1 \to n$ so $n \to n-1$

$$\therefore \quad 2M\mu_n \geqslant 1 + \frac{(n-1)M}{n}\mu_n - \frac{(n-1)M}{n}$$

$$\left(2M - \frac{n-1}{n}M\right)\mu_n \geqslant 1 - \frac{(n-1)M}{n}$$

$$\left(\frac{2n}{n} - \frac{n-1}{n}\right)M\mu_n \geqslant 1 - \frac{(n-1)M}{n}$$

$$\frac{n+1}{n}M\mu_n \geqslant 1 - \frac{(n-1)M}{n}$$

$$\boxed{\mu_n \geqslant \frac{n}{(n+1)M} - \frac{n-1}{n+1}}$$

as required.

Fin!

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