12/ list of n letters

a letter A $\qquad a \geq 2, \ b \geq 2$

b letter B

ie $a+b=n$. Now define random variables:

$$X_1 = \begin{cases} 1 & \text{First letter A} \\ 0 & \text{otherwise} \end{cases}$$

$$X_k = \begin{cases} 1 & (k-1)\text{th letter is B, } k\text{th is A} \\ 0 & \text{otherwise} \end{cases} \qquad 2 \leq k \leq n$$

$$S = \sum_{i=1}^{n} X_i$$

(i) $$E[X_1] = \underset{\substack{\uparrow \\ \text{probability} \\ \text{first letter is} \\ \text{A}}}{\frac{a}{n}} \times 1 + \underset{\substack{\uparrow \\ \text{probability} \\ \text{first letter is} \\ \text{not A}}}{\frac{b}{n}} \times 0 = \boxed{\frac{a}{n}}$$

$$E[X_{i \neq 1}] = \frac{ab}{n(n-1)} \times 1 + \left(1 - \frac{ab}{n(n-1)}\right) \times 0 = \boxed{\frac{ab}{n(n-1)}}$$

[With B at position $k-1$ and A at position k, this means there are $a-1$ A's left and $b-1$ B's left. So # perms are $\frac{(n-2)!}{(a-1)!(b-1)!}$ of the n letters, sans A and B.

The total permutations of the n letters are $\frac{n!}{a!\,b!} = \frac{n(n-1)(n-2)!}{ab\,(a-1)!(b-1)!}$

So probability of the "BA at position $k-1, k$ situation"

is $$\frac{\frac{(n-2)!}{(a-1)!(b-1)!}}{\frac{n(n-1)(n-2)!}{ab\,(a-1)!(b-1)!}} = \boxed{\frac{ab}{n(n-1)}}$$]

$$\therefore \quad E[S] = E[X_1] + \sum_{i=2}^{n} E[X_i] \quad \leftarrow n-1 \text{ terms}$$

$$= \frac{a}{n} + \frac{ab}{n(n-1)}(n-1)$$

$$= \boxed{\frac{a}{n}(1+b)} \quad \text{as required.}$$

(ii) a) $E[X_1 X_j]$ only has non-zero terms when first letter is A, $(j-1)^{th}$ is B and j^{th} is A

So $n-3$ terms with $a-2$ A's and $b-1$ B's to randomly permute. The probability of this situation is $\therefore$

$$\frac{\frac{(n-3)!}{(a-2)!(b-1)!}}{\frac{n!}{a!b!}} = \frac{a(a-1)b}{n(n-1)(n-2)}$$

Since $X_1 X_j = 1 \times 1$ X_1 and X_j are both non zero

$$\Rightarrow \boxed{E[X_1 X_j] = \frac{a(a-1)b}{n(n-1)(n-2)}} \quad \text{as required}$$

(ie $E[X] = \sum_i x_i P(x_i)$)

b) Similarly $E[X_i X_j]$ has non zero terms (and $X_i X_j = 1$) when $(i-1)^{th}$ is B, i is A, $(j-1)^{th}$ is B and j^{th} is A.

[Note question tells us that $i, j \neq 1$]

Probability of this is

$$\frac{\frac{(n-4)!}{(a-2)!(b-2)!}}{\frac{n!}{a!b!}} = \boxed{\frac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)}}$$

So $E[X_i X_j] = \dfrac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)}$ $\quad i,j \neq 1$

$\therefore \displaystyle\sum_{j=i+2}^{n} E[X_i X_j] = \dfrac{(n-i-1)\, a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)}$

$[\; n = N + i + 2 - 1 \quad \therefore \;\#\text{ terms } N = n-i-1 \;]$

$\uparrow$ since include term $i+2$

$\therefore \displaystyle\sum_{i=2}^{n-2}\sum_{j=i+2}^{n} E[X_i X_j] = \dfrac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)} \sum_{i=2}^{n-2}(n-i-1)$

$\displaystyle\sum_{i=2}^{n-2}(n-i-1) = \sum_{q=1}^{n-3}(n-2-q) = (n-2)(n-3) - \sum_{q=1}^{n-3} q$

$[\text{ let } i = q+1 \;\therefore\; q = i-1 \,]$ $\quad \left[\displaystyle\sum_{q=1}^{N} q = \tfrac{1}{2}N(N+1)\right]$

$= (n-3)(n-2) - \tfrac{1}{2}(n-3)(n-2)$

$= \dfrac{(n-3)}{2}\left(2n-4-n+2\right)$

$= \dfrac{(n-3)}{2}(n-2)$

$\therefore \displaystyle\sum_{i=2}^{n-2}\sum_{j=i+2}^{n} E[X_i X_j] = \boxed{\dfrac{a(a-1)b(b-1)}{2n(n-1)}}$ as required

c) $V[S] = E[S^2] - \left(E[S]\right)^2$

$S^2 = \left(\displaystyle\sum_{i=1}^{n} X_i\right)^2 = \sum_{i=1}^{n} X_i^2 + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} 2X_i X_j$

$$\therefore \quad E[S^2] = E\left[\sum_{i=1}^{n} X_i^2\right] + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} 2E[X_iX_j]$$

Now non zero values of X_i^2 are the same fraction of n as X_i so $E\left[\sum_{i=1}^{n} X_i^2\right] = E[S] = \frac{a(b+1)}{n}$

Now $\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} 2E[X_iX_j] = \frac{a(a-1)b(b+1)}{n(n-1)}$ {According to the official answers}

whereas above we showed $\sum_{i=2}^{n-2}\sum_{j=i+2}^{n} E[X_iX_j] = \frac{a(a-1)b(b-1)}{2n(n-1)}$

ie $b \to b+1$ makes this correspondence happen.

* [Proof required]

So if $E[S^2] = \frac{a(b+1)}{n} + \frac{a(a-1)b(b+1)}{n(n-1)}$

$$\therefore \quad V[S] = \frac{a(b+1)}{n} + \frac{a(a-1)b(b+1)}{n(n-1)} - \frac{a^2(b+1)^2}{n^2}$$

$$= \frac{a(b+1)n(n-1)}{n^2(n-1)} + \frac{na(a-1)b(b+1)}{n^2(n-1)} - \frac{a^2(b+1)^2(n-1)}{n^2(n-1)}$$

$$= \frac{a(b+1)}{n^2(n-1)}\Big(n(n-1) + n(a-1)b - a(b+1)(n-1)\Big)$$

Now $n = a+b$ so $(\ldots) = (a+b)(a+b-1) + (a+b)(a-1)b$
$- a(b+1)(a+b-1)$

$$= (a+b)(\cancel{a}+\cancel{b}-1+\cancel{ab}-\cancel{b}-\cancel{ab}-\cancel{a}) + a(b+1)$$

$$= (a+b)(-1) + ab + a$$

$$= -\cancel{a} - b + ab + \cancel{a}$$

$$= b(a-1)$$

$$\therefore \boxed{V[S] = \frac{a(b+1)b(a-1)}{n^2(n-1)}} \quad \text{as required.}$$