STEP III 2013

1/ $\boxed{t = \tan\frac{1}{2}x}$ $\quad \frac{dt}{dx} = \frac{1}{2}\sec^2\frac{1}{2}x$ $\quad$ Now $1+\tan^2\frac{1}{2}x = \sec^2\frac{1}{2}x$

$\therefore \frac{dt}{dx} = \frac{1}{2}(1+\tan^2\frac{1}{2}x) \quad \therefore \boxed{\frac{dt}{dx} = \frac{1}{2}(1+t^2)}$ as required

$$\sin x = \sin(\tfrac{1}{2}x + \tfrac{1}{2}x) = 2\sin\tfrac{1}{2}x\cos\tfrac{1}{2}x$$
$$= 2\tan\tfrac{1}{2}x\cos^2\tfrac{1}{2}x$$

$$\sec^2\tfrac{1}{2}x = \frac{1}{\cos^2\frac{1}{2}x} \quad \therefore \cos^2\tfrac{1}{2}x = \frac{1}{\sec^2\frac{1}{2}x} = \frac{1}{1+\tan^2\frac{1}{2}x}$$

$\therefore \sin x = \dfrac{2\tan\frac{1}{2}x}{1+\tan^2\frac{1}{2}x} \Rightarrow \boxed{\sin x = \dfrac{2t}{1+t^2}}$ as required

Consider $I = \displaystyle\int_0^{\pi/2} \frac{1}{1+a\sin x}\,dx$ $\quad$ let $t = \tan\frac{1}{2}x$

From above $\dfrac{dt}{dx} = \frac{1}{2}(1+t^2) \therefore dx = \dfrac{2}{1+t^2}\,dt$

$\sin x = \dfrac{2t}{1+t^2}$ $\quad$ when $x=0,\ t=0$

$x=\frac{\pi}{2},\ t=\tan\frac{\pi}{4}=1$

$$\therefore I = \int_0^1 \frac{1}{1+\frac{2at}{1+t^2}}\,\frac{2}{1+t^2}\,dt$$

$$I = 2\int_0^1 \frac{1}{1+t^2+2at}\,dt$$

Now $\boxed{\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c}$ [Standard integral]

So need to write $1+t^2+2at$ in this form.

Completing the square: $t^2+2at+1 = (t+a)^2-a^2+1$

$$\text{So } I = 2\int_0^1 \frac{1}{(t+a)^2+(1-a^2)}\,dt$$

let $z = t+a \quad \therefore \quad I = 2\displaystyle\int_a^{1+a} \frac{1}{z^2+(1-a^2)}\,dz$

$$\therefore \quad I = 2\left[\frac{1}{\sqrt{1-a^2}} \tan^{-1}\left(\frac{z}{\sqrt{1-a^2}}\right)\right]_a^{1+a}$$

[So this explains requirement $0 < a < 1$]

ie $\sqrt{1-a^2}$ only real if $|a| \leq 1$

$$I = \frac{2}{\sqrt{1-a^2}}\left\{\tan^{-1}\left(\frac{1+a}{\sqrt{1-a^2}}\right) - \tan^{-1}\left(\frac{a}{\sqrt{1-a^2}}\right)\right\}$$

Now $\boxed{\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}}$

So $\quad A \pm B = \tan^{-1}\left(\dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}\right)$

let $\tan A = x$

$\quad\;\; \tan B = y$

$\therefore \boxed{\tan^{-1} x \pm \tan^{-1} y = \tan^{-1}\left(\dfrac{x \pm y}{1 \mp xy}\right)}$

$$\therefore \tan^{-1}\left(\frac{1+a}{\sqrt{1-a^2}}\right) - \tan^{-1}\left(\frac{a}{\sqrt{1-a^2}}\right) = \tan^{-1}\left(\frac{\frac{1+a}{\sqrt{1-a^2}} - \frac{a}{\sqrt{1-a^2}}}{1 + \left(\frac{1+a}{\sqrt{1-a^2}}\right)\left(\frac{a}{\sqrt{1-a^2}}\right)}\right)$$

$$= \tan^{-1}\left(\frac{1+a-a}{\sqrt{1-a^2}\left(1 + \frac{a+a^2}{1-a^2}\right)}\right)$$

$$= \tan^{-1}\left(\frac{1-a^2}{\sqrt{1-a^2}\left(1-a^2+a+a^2\right)}\right)$$

$$= \tan^{-1}\left(\frac{1-a^2}{\sqrt{1-a^2}\,(1+a)}\right)$$

$$= \tan^{-1}\left(\frac{(1+a)(1-a)}{\sqrt{1+a}\,\sqrt{1-a}\,(1+a)}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{1-a}}{\sqrt{1+a}}\right)$$

$\therefore \boxed{\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+a\sin x}\,dx = \frac{2}{\sqrt{1-a^2}} \tan^{-1}\left(\frac{\sqrt{1-a}}{\sqrt{1+a}}\right)}$ as required

[$0 < a < 1$]

Now let $\quad I_n = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{2+\sin x}\, dx \qquad n \geq 0$

$$I_{n+1} + 2I_n = \int_0^{\frac{\pi}{2}} \frac{\sin^{n+1} x + 2\sin^n x}{2+\sin x}\, dx$$

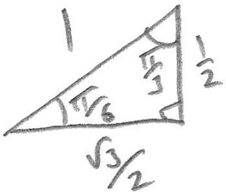
$$= \int_0^{\frac{\pi}{2}} \frac{\sin^n x\,(\sin x + 2)}{\sin x + 2}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^n x\, dx$$

$$I_0 = \int_0^{\frac{\pi}{2}} \frac{1}{2+\sin x}\, dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{1}{2}\sin x}\, dx$$

using the first result: $\quad I_0 = \dfrac{1}{2} \times \dfrac{2}{\sqrt{1-\left(\frac{1}{2}\right)^2}} \tan^{-1}\left(\dfrac{\sqrt{1-\frac{1}{2}}}{\sqrt{1+\frac{1}{2}}}\right)$

$$I_0 = \frac{1}{\sqrt{\frac{3}{4}}} \tan^{-1}\left(\frac{\sqrt{\frac{1}{2}}}{\sqrt{\frac{3}{2}}}\right)$$



$$I_0 = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$I_0 = \frac{2}{\sqrt{3}} \frac{\pi}{6}$$

$$\boxed{I_0 = \frac{\pi}{3\sqrt{3}}}$$

$$\therefore \quad I_1 + 2I_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$$

$$\therefore \quad \boxed{I_1 = \frac{\pi}{2} - \frac{2\pi}{3\sqrt{3}}}$$

$$I_2 + 2I_1 = \int_0^{\frac{\pi}{2}} \sin x\, dx = \left[-\cos x\right]_0^{\pi/2} = \left(-\cos\tfrac{\pi}{2}\right) - \left(-\cos 0\right)$$

$$= 1$$

$$\therefore \quad \boxed{I_2 = 1 - \pi + \frac{4\pi}{3\sqrt{3}}}$$

$$\therefore \quad I_3 + 2I_2 = \int_0^{\frac{\pi}{2}} \sin^2 x \, dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos 2x)\,dx$$

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(\left(\frac{\pi}{2}-0\right)-(0-0)\right)$$

$$= \pi/4$$

$$\therefore \quad I_3 = \pi/4 - 2\left(1-\pi + \frac{4\pi}{3\sqrt{3}}\right)$$

$$I_3 = \pi\left(\frac{1}{4}+2-\frac{8}{3\sqrt{3}}\right) - 2$$

$$\boxed{I_3 = \pi\left(\frac{9}{4}-\frac{8\sqrt{3}}{9}\right) - 2}$$