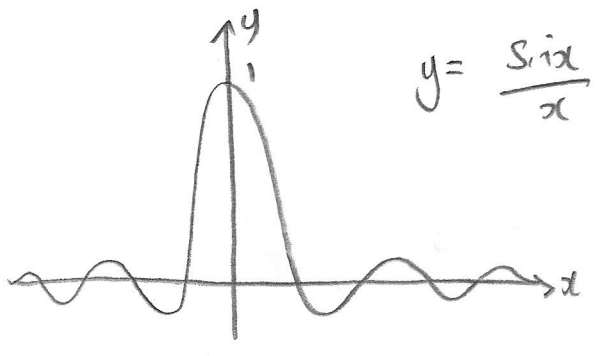


Integration of $\frac{\sin x}{x}$ and $\frac{\sin^2 x}{x^2}$

(via two neat tricks...)



EVEN FUNCTION

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\text{so } \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$$

which means $\frac{\sin x}{x} = 1$ when $x=0$.

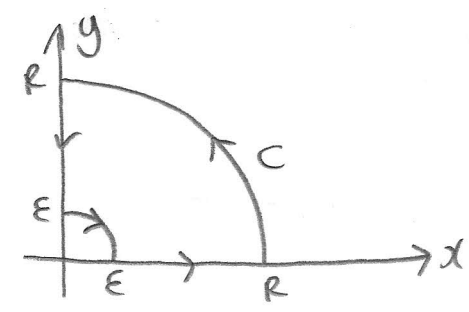
let $I = \int_{-\infty}^{\infty} \frac{\sin x}{x} dx$

Since $\frac{\sin x}{x}$ is even,

$$I = 2 \int_0^{\infty} \frac{\sin x}{x} dx$$

Consider plane.

$\oint_C \frac{e^{iz}}{z} dz$ is a contour integral in the complex plane.



Now $\frac{e^{iz}}{z} = \frac{\cos z + i \sin z}{z}$

So has a pole at $z=0$ ($\frac{\cos(0)}{0}$)

This is why we have the ϵ path to avoid this pole.

By Cauchy's Theorem: within C.

$$\oint_C \frac{e^{iz}}{z} dz = 0$$

Since no poles

$$\oint_C \frac{e^{iz}}{z} dz = \int_{\epsilon}^R \frac{\cos x}{x} dx + \int_0^{\frac{\pi}{2}} \frac{e^{iR(\cos \theta + i \sin \theta)}}{R e^{i\theta}} i R e^{i\theta} d\theta + \dots$$

$[z = x + iy = R e^{i\theta} \text{ (or } \epsilon e^{i\theta}) = (\cos \theta + i \sin \theta) \times R, \text{ or } \epsilon]$

See abv.

$$+ \int_R^\infty \frac{i \sin y}{iy} (-idy) + \int_{\frac{\pi}{2}}^0 \frac{e^{i\epsilon(\cos\theta + i\sin\theta)}}{\epsilon e^{i\theta}} i\epsilon e^{i\theta} d\theta$$

$$\therefore 0 = \int_\epsilon^R \frac{\cos x}{x} dx + i \int_\epsilon^R \frac{\sin y}{y} dy$$

$$+ i \int_0^{\frac{\pi}{2}} \left(e^{-R\sin\theta} e^{iR\cos\theta} - e^{-\epsilon\sin\theta} e^{i\epsilon\cos\theta} \right) d\theta$$

Now $\lim_{\substack{R \rightarrow \infty \\ \epsilon \rightarrow 0}} \oint_C \frac{e^{iz}}{z} dz$ is still zero (!)

$$\text{So } 0 = \int_0^\infty \frac{\cos x}{x} dx + i \int_0^\infty \frac{\sin y}{y} dy + i \int_0^{\frac{\pi}{2}} (-i) d\theta$$

$$\text{Since } \lim_{R \rightarrow \infty} \left(e^{-R\sin\theta} e^{iR\cos\theta} \right) = 0$$

$$\lim_{\epsilon \rightarrow 0} \left(e^{-\epsilon\sin\theta} e^{i\epsilon\cos\theta} \right) = 1$$

$$\therefore \text{Since } \int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2}$$

$$i\frac{\pi}{2} = \int_0^\infty \frac{\cos x}{x} dx + i \int_0^\infty \frac{\sin y}{y} dy$$

$$\Rightarrow \int_0^\infty \frac{\sin y}{y} dy = \frac{\pi}{2}$$

$$\text{and } \int_0^\infty \frac{\cos x}{x} dx = 0$$

obvious given
 $\frac{\cos x}{x}$ is odd.

$$\therefore \boxed{I = \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi}$$

Now consider
$$J(a) = \int_{-\infty}^{\infty} \frac{\sin^2(ax)}{x^2} dx$$

$$\frac{dJ}{da} = \int_{-\infty}^{\infty} \frac{1}{x^2} \frac{d}{da} (\sin^2 ax) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{x^2} 2 \sin ax \cos ax \cdot x dx$$

$$= \int_{-\infty}^{\infty} \frac{2 \sin ax \cos ax}{x} dx$$

$$= \int_{-\infty}^{\infty} \frac{\sin 2ax}{x} dx$$

$$[\sin 2\theta = 2 \sin \theta \cos \theta]$$

let $z = 2ax \quad \therefore dx = \frac{dz}{2a}$
 $x = \frac{z}{2a}$

$$\therefore \frac{dJ}{da} = \int_{-\infty}^{\infty} \frac{\sin z}{z/2a} \frac{dz}{2a} = \int_{-\infty}^{\infty} \frac{\sin z}{z} dz$$

Using the result above: $\frac{dJ}{da} = \pi$

$$\therefore J = \pi a + C$$

Now when $a=0$, $J=0 \quad \therefore C=0$

So $J(a) = \pi a$

$$\therefore \int_{-\infty}^{\infty} \frac{\sin^2 ax}{x^2} dx = \pi a$$

eg
$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \pi$$

So what about $\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^n dx$ you may ask?

well sadly it gets a bit complicated when $n \neq 1, 2$.

According to [mathworld.wolfram.com/sinefunction.html](http://mathworld.wolfram.com/SineFunction.html)

$$\left[\text{Sinc}(x) = \frac{\sin x}{x} \right]$$

there is a formula, but it is rather complicated!

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi \quad \int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \pi$$

$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^3 dx = \frac{3\pi}{4} \quad \int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^4 dx = \frac{2\pi}{3}$$

$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^5 dx = \frac{115\pi}{192}$$

[Idea consider $\frac{d^n S}{da^n}$ where $S(a) = \int_{-\infty}^{\infty} \frac{\sin(ax)}{x^n} dx$

write in terms of $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$ and then integrate w.r.t a n times \dots]

↑
An exercise for the keen student!

AT. May 2016.