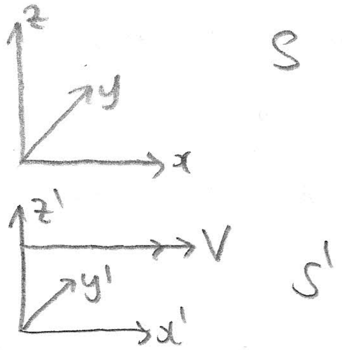
1/



$\boxed{\text{Lorentz transform}}$ relates distance x and time t between 'Static' frame S and moving frame S', which moves // x at velocity V

$$\boxed{t = \gamma\left(t' + \frac{Vx'}{c^2}\right)} \qquad \boxed{\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}}$$

Now if clock is stationary in S', $x' = 0$

$\therefore \quad t = \gamma t'$

let $t' = \frac{1}{2}t \quad \therefore \quad t = \gamma/_2 \, t \Rightarrow \boxed{\gamma = 2}$

ie "moving clock runs at half the rate of an equivalent clock in S"

$$\Rightarrow \quad 1 - \frac{v^2}{c^2} = \frac{1}{4} \quad \Rightarrow \quad \frac{v^2}{c^2} = \frac{3}{4}$$

$$\Rightarrow \quad \boxed{V = \frac{\sqrt{3}}{2}c} \qquad \approx \boxed{0.866c}$$

2/ A metre stick travels at speed V relative to frame S // to the x direction. In a frame S' where the metre stick is stationary

$$\boxed{x' = \gamma(x - Vt)} \quad \text{from the Lorentz Transform}$$

Now $x' = 1$ metre and let stick length be measured in S when $t = 0$.

Here $\quad 1 = \gamma x \quad \therefore \quad x = \frac{1}{\gamma} \quad$ ie 'lengths contract'

Now if $x = 0.9\,m$

$$\gamma = \frac{10}{9}$$

$$\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \frac{10}{9}$$

$$1 - \frac{v^2}{c^2} = \frac{81}{100}$$

$$\frac{v^2}{c^2} = 1 - \frac{81}{100}$$

$$\frac{v^2}{c^2} = \frac{19}{100}$$

$$\boxed{V = \frac{\sqrt{19}}{10}c}$$

$$\approx \boxed{0.436c}$$

3/ let $t' = 26\,ns$ be lifetime of the π meson in its rest frame.

Hence taking the Earth to be a frame S and the meson to be at rest in a frame S' moving at speed $V = 0.95c$ relative to S in the x direction

Lorentz transform $\Rightarrow$ $t = \gamma(t' + Vx'/c^2)$ $\quad x' = 0$

So $t = (1-0.95^2)^{-\frac{1}{2}} \times 26\,ns$

$t = 3.20 \times 26\,ns$

$\boxed{t = 83.3ns}$

So meson will travel $\simeq$ $0.95c \times 83.3ns \simeq \boxed{23.7m}$

before it decays.

4/ Atomic clock measures time $t' = 3600s$ on-board a jet moving at $400\,ms^{-1}$

on the ground, the equivalent time interval is $t = \gamma t'$

(Note $x' = 0$ as it is assumed the clock is at rest relative to the S' frame co-moving with the aircraft)

$$t = \left(1 - \left(\frac{\overset{v}{400}}{\underset{c}{2.99 \times 10^8}}\right)^2\right)^{-\frac{1}{2}} \times 3600$$

Now calculator precision problems working this out directly

So let $t = 3600 + \delta t$, and since $v^2/c^2 \ll 1$

use a single term Binomial expansion

$$t \simeq t'\left(1 + \left(-\tfrac{1}{2}\right)\left(-\frac{v^2}{c^2}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{v^2}{c^2}\right)^2}{2} + \dots\right)$$

$$\simeq t'\left(1 + \frac{v^2}{2c^2}\right) \quad \therefore \quad \boxed{\delta t \simeq \frac{t' v^2}{2c^2}}$$

$$\therefore \delta t = \frac{3600 \times 400^2}{2 \times (2.998 \times 10^8)^2} \simeq \boxed{3.2ns}$$

5/ observed lifetime of muons is $\boxed{\tau = \gamma \tau'}$

where τ' is lifetime in rest frame of the muon

If the muon travels at speed $0.95c$

$\tau = (1 - 0.95^2)^{-\frac{1}{2}} \tau'$ $\qquad \tau' = 2.2\,\mu s$

$\therefore \boxed{\tau = 7.05\,\mu s}$

# of muons after time t is $\qquad N = N_0 e^{-t/\tau}$

So if muons travel distance x, $\quad t = x/0.95c$

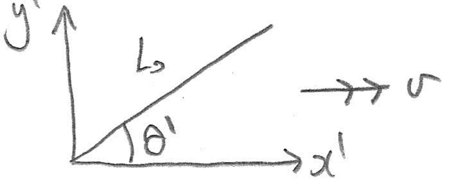
$$\therefore \frac{N}{N_0} = \exp\left(\frac{-x}{0.95c \times 7.05 \times 10^{-6}}\right)$$

let $x = 3000\,m$ $\quad \therefore \frac{N}{N_0} = \exp\left(\frac{-3000}{0.95 \times 2.998 \times 10^{8} \times 7.05 \times 10^{-6}}\right)$

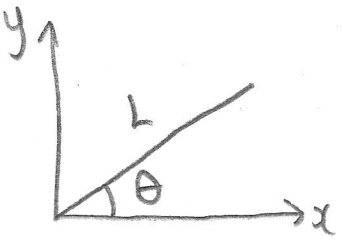
$= 0.224$

ie only about $\boxed{22\%\ \text{left}}$

6/ In rest frame of rod (S'), which moves at speed v w.r.t to a frame S.



The rod is photographed in S and looks like



using Lorentz transform $\begin{cases} y' = y \\ x' = \gamma(x - vt) \end{cases}$

let photograph occur when $t = 0$ (It is important that this is for the S frame)

So $L_0 \sin\theta' = L \sin\theta$ ① (y)

$L_0 \cos\theta' = \gamma L \cos\theta$ ② (x)

So ①/②: $\boxed{\tan\theta = \gamma \tan\theta'}$ $\qquad$ Since $\gamma \geq 1$ and $0 < \theta < 90°$

$\frac{d}{d\theta}\tan\theta > 0 \Rightarrow \boxed{\theta > \theta'}$

ie rod is rotated upwards in S.

$① ^2 + ② ^2$: $L^2(\sin^2\theta + \gamma^2\cos^2\theta) = L_0^2(\sin^2\theta' + \cos^2\theta') = L_0^2$

$$\therefore \boxed{L = \frac{L_0}{\sqrt{\sin^2\theta + \gamma^2\cos^2\theta}}}$$

Now it is not clear whether $L > L_0$ or $L < L_0$ here since $\sin^2\theta$ and $\cos^2\theta < 1$

$\therefore$ Instead use, $L_0^2\cos^2\theta' = \gamma^2 L^2 \cos^2\theta$

$$\therefore \quad L_0^2 \sec^2\theta = \gamma^2 L^2 \sec^2\theta'$$

$$L_0^2(1 + \tan^2\theta) = \gamma^2 L^2 (1 + \tan^2\theta')$$

Now $\tan\theta = \gamma \tan\theta'$ so

$$L_0^2(1 + \gamma^2\tan^2\theta') = \gamma^2 L^2 (1 + \tan^2\theta')$$

$$L = L_0\sqrt{\frac{(1 + \gamma^2\tan^2\theta')}{\gamma^2(1 + \tan^2\theta')}}$$

$$L = L_0\sqrt{\frac{1/\gamma^2 + \tan^2\theta'}{1 + \tan^2\theta'}}$$

$$1/\gamma^2 = 1 - v^2/c^2$$

So $$L = L_0\sqrt{\frac{1 + \tan^2\theta' - v^2/c^2}{1 + \tan^2\theta'}}$$

$$\boxed{L = L_0\sqrt{1 - \frac{v^2}{c^2}\cos^2\theta'}}$$

$1 + \tan^2\theta' = \frac{1}{\cos^2\theta'}$

Since $\frac{v^2\cos^2\theta'}{c^2} > 0$

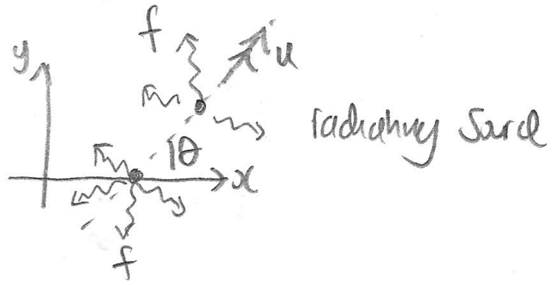
$\Rightarrow \boxed{L < L_0}$ ie contracts

④

7/ Relativistic Doppler Shift formula

$$\boxed{f = \frac{f'}{\gamma\left(1 + \frac{u\cos\theta}{w}\right)}}$$

w = wave speed
f' = frequency in rest frame of source.



let $w = c$ $\therefore \lambda f = c$
$\lambda' f' = c$

So $$\frac{c}{\lambda} = \frac{c}{\lambda'\gamma\left(1 + u\cos\theta/c\right)}$$

$$\therefore \boxed{\lambda = \lambda'\gamma\left(1 + u\cos\theta/c\right)}$$

Now when $\theta = 180°$ (i.e. source moving towards observer at $x=0, y=0$) $\lambda_G = 500$ nm.

when $\theta = 0°$, $\lambda_R = 700$ nm

For source (the relativistic taxi), $u = \frac{c}{10}$ $\therefore \gamma = (1 - 0.1^2)^{-\frac{1}{2}}$

So $\lambda_G = \lambda'\gamma(1 - u/c)$
$\lambda_R = \lambda'\gamma(1 + u/c)$

$$\therefore \frac{\lambda_R}{\lambda_G} = \frac{1 + u/c}{1 - u/c} \Rightarrow \left(1 - \frac{u}{c}\right)\frac{\lambda_R}{\lambda_G} = 1 + u/c$$

$$u/c\left(1 + \frac{\lambda_R}{\lambda_G}\right) = \frac{\lambda_R}{\lambda_G} - 1 \quad \therefore u = c\frac{\left(\frac{\lambda_R}{\lambda_G} - 1\right)}{\frac{\lambda_R}{\lambda_G} + 1}$$

$$\boxed{u = \frac{c(\lambda_R - \lambda_G)}{\lambda_R + \lambda_G}} \quad \therefore u = \frac{200}{1200}c = \boxed{\frac{c}{6}}$$

$$\boxed{u \simeq 0.17c}$$

5

$$\therefore \lambda' = \frac{\lambda_G}{\gamma(1-u/c)}$$

$$u = -\frac{1}{6}c$$

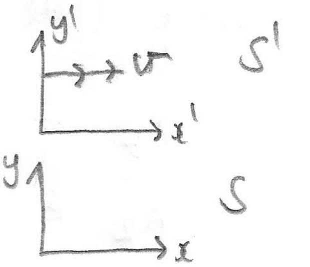
$$\gamma = \left(1 - \frac{1}{36}\right)^{-\frac{1}{2}} = \frac{6}{\sqrt{35}}$$

$$\therefore \lambda' = \frac{\lambda_G}{\frac{6}{\sqrt{35}} \times \frac{5}{6}}$$

$$\lambda' = \lambda_G \times \frac{\sqrt{35}}{5}$$

$$\lambda' = 500\,\text{nm} \times 1.183$$

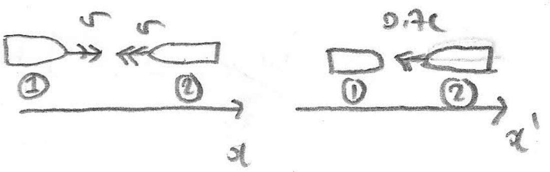
$$= \boxed{592\,\text{nm}}$$



8/ let S' be rest frame of spaceship 1
S be the Earth frame

Lorentz transform of velocity

$$\boxed{u_x' = \frac{u_x - v}{1 - u_x v/c^2}}$$



For brevity let all velocities be ∥ x and x' directions

$u_1' = 0 \quad u_2' = -0.7c$

$u_1 = v \quad u_2 = -v$

$$-0.7c = \frac{-2v}{1 + v^2/c^2}$$

$$\left(1 + v^2/c^2\right)0.7c = 2v$$

$$0.7\frac{v^2}{c} - 2v + 0.7c = 0$$

$$0.7\left(\frac{v}{c}\right)^2 - 2\left(\frac{v}{c}\right) + 0.7 = 0$$

$$\frac{v}{c} = \frac{2 \pm \sqrt{4 - 4(0.7)^2}}{1.4}$$

$$\frac{v}{c} = \frac{2 \pm 2\sqrt{1 - 0.7^2}}{1.4}$$

$$\frac{v}{c} = \frac{1}{0.7}\left(1 \pm \sqrt{1 - 0.7^2}\right)$$

$$\frac{v}{c} = 0.408,\ 2.45$$

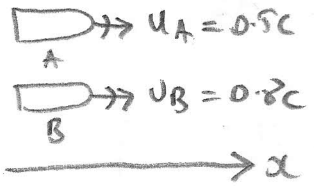
Now $v \leq c$ so only solution is $\boxed{v = 0.408c}$

$\therefore$ in S frame $\boxed{\begin{aligned} u_1 &= 0.408c \\ u_2 &= -0.408c \end{aligned}}$

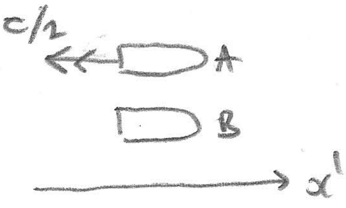
9/ $U_A = 0.5c$
$U_B = 0.8c$

Define S' frame to be moving towards the Earth at $v = 0.8c$, ie rest frame of B. Let x point towards the Earth.

So $u_A' = \dfrac{U_A - v}{1 - u_A v/c^2} = \dfrac{0.5c - 0.8c}{1 - 0.5 \times 0.8\, c^2/c^2}$



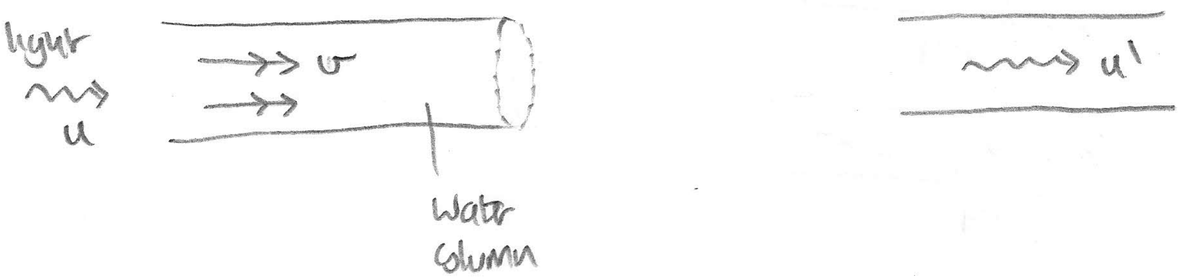
$$u_A' = \frac{-0.3c}{1 - 0.4}$$



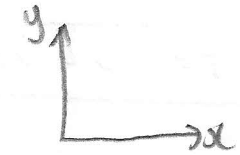
$$u_A' = \frac{-0.3c}{0.6}$$

$u_A' = \boxed{-c/2}$ away from Earth relative to B.

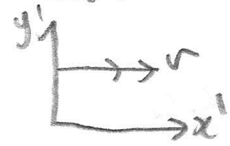
10/



lab frame S



S' frame co-moving with water



velocity transformations $\quad u = \dfrac{u' + v}{1 + \frac{u'v}{c^2}}$

$u' = \dfrac{c}{n}$ (n = refractive index of water)

$\therefore \quad u = \left(v + \frac{c}{n}\right)\left(1 + \frac{vc}{nc^2}\right)^{-1}$

$u = \left(v + \frac{c}{n}\right)\left(1 + \frac{v}{nc}\right)^{-1}$ Now $\frac{v}{nc} \ll 1$

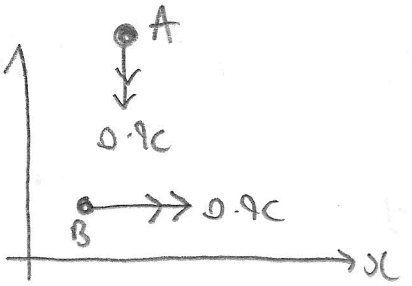
$\therefore \quad u = \left(v + \frac{c}{n}\right)\left(1 - \frac{v}{nc} + \frac{(-1)(-2)\left(\frac{v}{nc}\right)^2}{2} + \dots\right)$

$u = v + \frac{c}{n} - \frac{v^2}{nc} - \frac{v}{n^2} + \dots\dots$

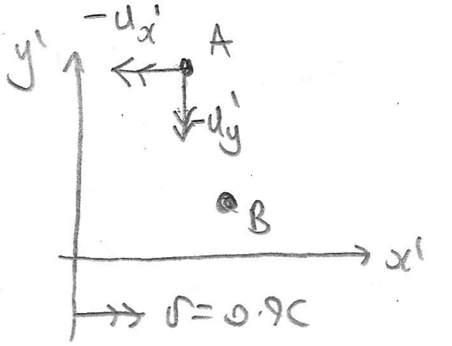
$u = \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right) + O\left(\frac{v^2}{c}\right)$

So $\boxed{u \simeq \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right)}$ as required.

11)



S frame



S' frame
(B is stationary)

velocity transforms
$$u_x' = \frac{u_x - v}{1 - u_x v/c^2}$$

$$u_y' = \frac{u_y}{\gamma(1 - u_x v/c^2)} \qquad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

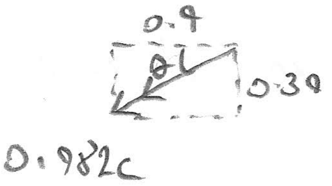
Now $u_x = 0 \quad u_y = -0.9c \quad v = 0.9c$

So $-u_x' = -0.9c \Rightarrow \boxed{u_x' = 0.9c}$

$$-u_y' = \frac{-0.9c}{(1-0.9^2)^{-\frac{1}{2}}} \Rightarrow \boxed{u_y' = 0.39c}$$

So from B, A moves at velocity $\sqrt{(u_x')^2 + (u_y')^2} = 0.982c$

at angle



$$\theta = \tan^{-1}\left(\frac{0.39}{0.9}\right)$$
$= \boxed{23.6°}$ from horizontal

12) Relativistic momentum $\boxed{\underline{p} = \gamma m \underline{u}}$ $\gamma = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}}$

velocity

$u = |\underline{u}|$

$$\boxed{\underline{f} = \frac{d}{dt}(\gamma m \underline{u})}$$

(Newton II: Force = rate of change of momentum)

$\therefore$ $\underline{f} = m\gamma\underline{a} + m\underline{u}\frac{d\gamma}{dt}$ if m = constant
mass

$\underline{a}$ is acceleration $\underline{a} = \frac{d\underline{u}}{dt}$

Now $$\frac{d\gamma}{dt} = \frac{d}{dt}\left(\left(1 - \frac{\underline{u}\cdot\underline{u}}{c^2}\right)^{-\frac{1}{2}}\right)$$

$$= -\frac{1}{2}\left(1 - \frac{\underline{u}\cdot\underline{u}}{c^2}\right)^{-3/2}\left(-\frac{1}{c^2}\right)\frac{d}{dt}(\underline{u}\cdot\underline{u})$$

$$= \frac{1}{2c^2}\gamma^3\left(\underline{u}\cdot\frac{d\underline{u}}{dt} + \frac{d\underline{u}}{dt}\cdot\underline{u}\right)$$

$$= \frac{\gamma^3}{c^2}(\underline{u}\cdot\underline{a})$$

$$\therefore \boxed{\underline{f} = m\gamma\underline{a} + \frac{m\gamma^3}{c^2}(\underline{u}\cdot\underline{a})\,\underline{u}} \quad (*)$$

Now when $\underline{f} \parallel \underline{u}$, it makes sense that $\underline{a} \parallel \underline{u}$

So write $(*)$ if $\underline{f} = F\hat{\underline{u}}$

$$F = m\gamma\frac{du}{dt} + \frac{m\gamma^3}{c^2}u^2\frac{du}{dt}$$

$$F = m\gamma^3\frac{du}{dt}\left(\frac{1}{\gamma^2} + \frac{u^2}{c^2}\right)$$

Now $\frac{1}{\gamma^2} = 1 - \frac{u^2}{c^2}$

$\therefore \quad \frac{1}{\gamma^2} + \frac{u^2}{c^2} = 1$

$\therefore \quad F = m\gamma^3 \frac{du}{dt}$

or $\boxed{F = m\left(1 - \frac{u^2}{c^2}\right)^{-3/2} \frac{du}{dt}}$

If F is a constant and $u=0$ when $t=0$

$$\frac{Ft}{m} = \int_0^u \left(1 - \frac{u^2}{c^2}\right)^{-3/2} du = \int_0^u \gamma^3 du$$

Now $\frac{d\gamma}{du} = -\frac{1}{2}\left(1 - \frac{u^2}{c^2}\right)^{-3/2}\left(-\frac{2u}{c^2}\right) = \gamma^3 \frac{u}{c^2}$

So $du = \frac{c^2}{u\gamma^3} d\gamma$

$\therefore \quad \gamma^3 du = \frac{c^2}{u} d\gamma$

Now $u = c\sqrt{1 - \frac{1}{\gamma^2}}$ (+ve version)

$u = \frac{c\sqrt{\gamma^2 - 1}}{\gamma}$

$\therefore \quad \frac{c^2}{u} = \frac{\gamma c}{\sqrt{\gamma^2-1}}$ So $\frac{Ft}{m} = c\int_1^\gamma \frac{\gamma}{\sqrt{\gamma^2-1}} d\gamma$

↑ when $u=0$, $\gamma = 1$

Now $\frac{d}{d\gamma}\sqrt{\gamma^2-1} = \frac{1}{2}(\gamma^2-1)^{-\frac{1}{2}} \times 2\gamma$

$$= \frac{\gamma}{\sqrt{\gamma^2-1}}$$

So $\int_1^\gamma \frac{\gamma}{\sqrt{\gamma^2-1}}\,d\gamma = \left[\sqrt{\gamma^2-1}\right]_1^\gamma$

$$= \sqrt{\gamma^2-1}$$

$$\therefore \quad \frac{Ft}{m} = c\sqrt{\gamma^2-1}$$

$$\therefore \quad \left(\frac{Ft}{mc}\right)^2 = \gamma^2-1$$

$$\gamma^2 = 1+\left(\frac{Ft}{mc}\right)^2 \qquad (\gamma>1)$$

so no -ve

$$\left(1-\frac{u^2}{c^2}\right)^{-1} = \frac{m^2c^2+F^2t^2}{m^2c^2}$$

$$1-\frac{u^2}{c^2} = \frac{m^2c^2}{m^2c^2+F^2t^2}$$

$$1-\frac{m^2c^2}{m^2c^2+F^2t^2} = \frac{u^2}{c^2}$$

$$\sqrt{\frac{m^2c^2+F^2t^2-m^2c^2}{m^2c^2+F^2t^2}}\;c = u$$

$$\therefore \quad u = c\sqrt{\frac{F^2t^2}{m^2c^2+F^2t^2}}$$

$$u/c = \sqrt{\frac{1}{\left(\frac{mc}{Ft}\right)^2 + 1}}$$

$$\boxed{u/c = \left(1 + \left(\frac{mc}{Ft}\right)^2\right)^{-\frac{1}{2}}}$$

In classical limit $u \ll c$ $\quad F = m\frac{du}{dt}$

So $\int_0^t \frac{F}{m} dt = \int_0^u du \Rightarrow \boxed{u = \frac{Ft}{m}}$

Now $u \ll c \Rightarrow u/c \ll 1$ which means

$1 + \left(\frac{mc}{Ft}\right)^2 \gg 1$ So $\frac{mc}{Ft} \gg 1$

$\therefore$ in classical limit $u/c \simeq \frac{Ft}{mc}$

$$\Rightarrow \boxed{u \simeq \frac{Ft}{m}}$$ ✓

13/ If an electron gains kinetic energy $k = 50\,kV \times 1.6 \times 10^{-19}$

$$\gamma = \left(1 - v^2/c^2\right)^{-\frac{1}{2}}$$

Relativistic	$k = (\gamma - 1)mc^2$
Classical	$k = \frac{1}{2}mv^2$

m = mass of an electron $= 9.109 \times 10^{-31}$ kg

(i) So $\gamma - 1 = \frac{k}{mc^2}$ $\quad \therefore \gamma = 1 + k/mc^2$

$$\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = 1 + \frac{k}{mc^2} \Rightarrow 1 - \frac{v^2}{c^2} = \left(1 + \frac{k}{mc^2}\right)^{-2}$$

$$\therefore \quad \frac{v^2}{c^2} = 1 - \left(1 + k/mc^2\right)^{-2}$$

$$\therefore \quad \boxed{v = c\sqrt{1 - \left(1 + k/mc^2\right)^{-2}}} \qquad (*)$$

Now in classical limit $v \ll c$

So $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = 1 + \frac{v^2}{2c^2} + \ldots$

$$\simeq 1 + \frac{v^2}{2c^2}$$

$$\therefore \quad (\gamma - 1)mc^2 \simeq \frac{v^2}{2c^2} mc^2 = \boxed{\tfrac{1}{2}mv^2} \checkmark$$

So if $k = \frac{1}{2}mv^2 \Rightarrow \quad v = \sqrt{\frac{2k}{m}}$

$$\therefore \quad v/c = \sqrt{\frac{2k}{mc^2}}$$

$$\frac{v}{c} = \sqrt{\frac{2 \times 50 \times 10^3 \times 1.602 \times 10^{-19}}{9.109 \times 10^{-31} \times (2.998 \times 10^8)^2}} \simeq \boxed{0.4423}$$

(classical expression)

Equivalent relativistic expression (*) yields

$$\boxed{v/c = 0.4127}$$

ie the classical expression yields a speed which is about

$$\frac{0.4423 - 0.4127}{0.4127} \times 100 \simeq \boxed{7\% \text{ larger}}$$

If the maximum distance travelled by an electron in the TV set $\simeq 0.5$ m. Difference in travel times $- t_{Rel} - t_{classical} = \Delta t$

$\Delta t \simeq \frac{0.5}{c}\left(\frac{1}{0.4127} - \frac{1}{0.4423}\right)$

$\simeq \boxed{0.27\ \text{ns}}$ $(1\ \text{ns} = 10^{-9}\ \text{s})$

If a TV scans at 50Hz and has the equivalent of HD digital resolution ie 1920 × 1080 pixels

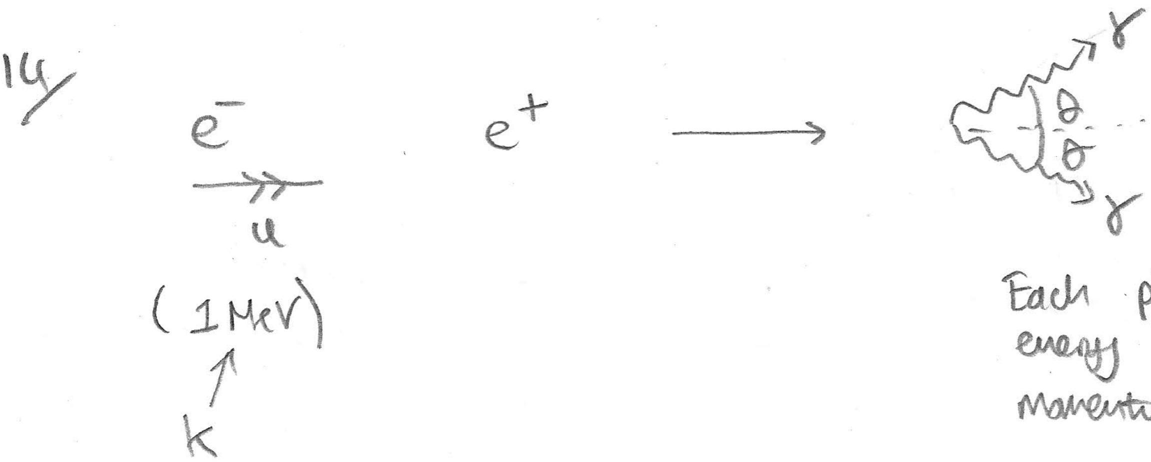
Time between pixels is $\simeq \delta t = \dfrac{1/50}{1920 \times 1080} = 9.6 \times 10^{-9}\ \text{s}$

ie $\simeq \boxed{10\ \text{ns}}$

So on this timescale the difference between the classical and relativistic travel times is less than an 'inter-pixel' distance by $\frac{1}{36}$ factor of

ie it is not likely to have a significant effect.

14/



Each photon has energy E_γ and momentum p_γ

let both electron and positron have mass $m = 9.109 \times 10^{-31}$ kg

By conservation of energy $k + 2mc^2 = 2E_\gamma$

So $E_\gamma = \frac{k}{2} + mc^2 = 0.5\ \text{MeV} + \dfrac{9.109 \times 10^{-31} \times (2.998 \times 10^{8})^2}{10^6 \times 1.609 \times 10^{-19}}$

$= \boxed{1.0088\ \text{MeV}}$

0.5088 MeV

Now using Energy - Momentum $\boxed{E^2 - p^2c^2 = m^2c^4}$ for the electron, we can find the total momentum before the collision (since the positron is at rest).

So $p = \dfrac{\sqrt{E^2 - m^2c^4}}{c}$

Now $E = mc^2 + k$ (total energy of electron)

So $E^2 = m^2c^4 + 2kmc^2 + k^2$

$\therefore$ $E^2 - m^2c^4 = k(k + 2mc^2)$

$\therefore$ $\boxed{p = \dfrac{\sqrt{k}\sqrt{k + 2mc^2}}{c}}$

Now sensible units are MeV for energy and MeV/c for momentum

$k = 1\text{ MeV}$

$mc^2 = 0.5088\text{ MeV}$

$\therefore$ $p = \sqrt{1 + 0.5088 \times 2}\ \text{MeV}/c = \boxed{1.420\text{ MeV}/c}$

Now by conservation of momentum

$\boxed{p = 2p_\gamma \cos\theta}$

Now $\boxed{E_\gamma = p_\gamma c}$ so $\theta = \cos^{-1}\left(\dfrac{p}{2p_\gamma}\right)$

$\theta = \cos^{-1}\left(\dfrac{pc}{2E_\gamma}\right)$

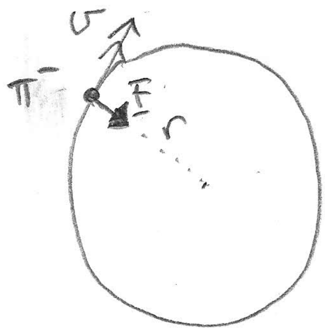
$\theta = \cos^{-1}\left(\dfrac{1.420}{1.0088}\right) = \boxed{45.3^\circ}$

15/ $K^0 \to \pi^+ + \pi^-$

$m_\pi = 140\ \mathrm{MeV}/c^2$

K^0 is initially at rest.

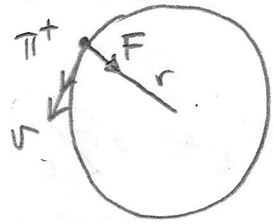
By conservation of energy $m_K c^2 = 2m_\pi c^2 + 2\mathcal{E} = 2E$
where $\mathcal{E}$ is the kinetic energy of each pion. (Assume the same!)



Force on pions due to magnetic field is

$$\underline{F} = q_\pi \underline{v} \times \underline{B}$$

ie $|\underline{F}| = \boxed{F = q_\pi v B}$



⊗ $\underline{B}$

Now $\underline{F} = m\gamma \underline{a} + m\gamma^3 \left(\frac{\underline{a}\cdot\underline{v}}{c^2}\right)\underline{v}$ (Newton II)

For circular motion $\underline{a}\cdot\underline{u} = 0$ So $\underline{F} = m\gamma \underline{a}$

Now $|\underline{a}| = v^2/r$

So $q_\pi v B = m_\pi \gamma v^2/r$

$q_\pi r B = \gamma m_\pi v$

$$\boxed{q_\pi r B = p_\pi}$$

Note $q_{\pi^+} = e$
$q_{\pi^-} = -e$

$e = 1.609\times10^{-19}\ \mathrm{C}$

By Energy-Momentum invariant for the pion

$$E^2 = m_\pi^2 c^4 + p_\pi^2 c^2$$

$$E = \sqrt{m_\pi^2 c^4 + e^2 r^2 B c^2}$$

$$m_\pi^2 c^4 = (140\ \text{MeV})^2$$

$$erBc = \frac{rBc}{10^6}\ \text{MeV}$$

$$= \frac{0.344 \times 2 \times 2.998 \times 10^8}{10^6}\ \text{MeV}$$

$$= 206.3\ \text{MeV}$$

So $E = \sqrt{140^2 + 206.3^2}\ \text{MeV}$

$$= 249.29\ \text{MeV}$$

$$\therefore\ m_k = 2E/c^2$$

$$= \boxed{498.6\ \text{MeV}/c^2}$$

From above $p_\pi = erB = \boxed{206.3\ \text{MeV}/c}$

Now $p = \gamma m v$

$$p^2 = \gamma^2 m^2 v^2$$

$$p^2 = \left(1 - \frac{v^2}{c^2}\right)^{-1} m^2 v^2$$

$$p^2 = \frac{c^2}{c^2 - v^2} m^2 v^2$$

$$p^2 (c^2 - v^2) = m^2 c^2 v^2$$

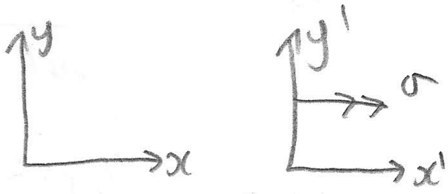
$$p^2 c^2 = v^2 (m^2 c^2 + p^2)$$

$$\therefore\ v^2 = \frac{p^2 c^2}{p^2 + m^2 c^2} \quad \therefore\ \boxed{v = \pm \frac{pc}{\sqrt{p^2 + m^2 c^2}}}$$

So $v_{\pi} = \pm \dfrac{206.3\,c}{\sqrt{206.3^2 + 140^2}} = \boxed{0.827c}$

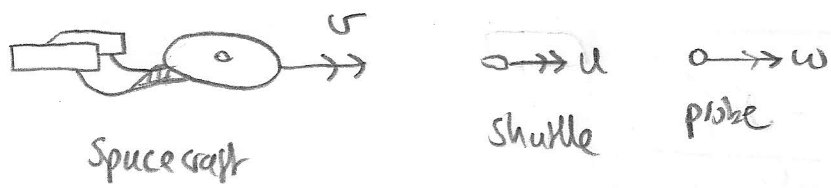
16/ Velocity transformation formula between S and S' frames

$$\boxed{u_x = \frac{u_x' + v}{1 + v u_x'/c^2}}$$



Earth frame

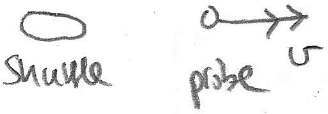
Spacecraft frame



$\therefore$ $\boxed{u = \dfrac{2v}{1 + v^2/c^2}}$ the speed of shuttle relative to Earth.

Shuttle frame

$\therefore$ $\boxed{w' = \dfrac{2v}{1+v^2/c^2}}$



So $\quad w = \dfrac{w' + v}{1 + w'v/c^2}$ [speed of probe relative to Earth]

$$\frac{w'v}{c^2} = \frac{2v^2}{c^2+v^2} \quad \therefore \quad 1 + \frac{w'v}{c^2} = \frac{c^2+v^2+2v^2}{c^2+v^2}$$

$$= \frac{3v^2+c^2}{c^2+v^2}$$

$$\therefore \quad w = \left(\frac{2v}{1+v^2/c^2} + v\right)\frac{c^2+v^2}{3v^2+c^2}$$

$$w = \left(\frac{2vc^2}{c^2+v^2} + \frac{v(c^2+v^2)}{c^2+v^2} \right) \left(\frac{c^2+v^2}{3v^2+c^2} \right)$$

$$w = \frac{2vc^2 + vc^2 + v^3}{3v^2 + c^2}$$

$$w = \frac{3vc^2 + v^3}{3v^2+c^2}$$

$$w = \frac{3v + v^3/c^2}{3v^2/c^2 + 1}$$

$$\boxed{w = v \left(\frac{3 + v^2/c^2}{3v^2/c^2 + 1} \right)}$$

Now in classical limit $v \ll c$

$$\Rightarrow \quad \boxed{w \to 3v}$$

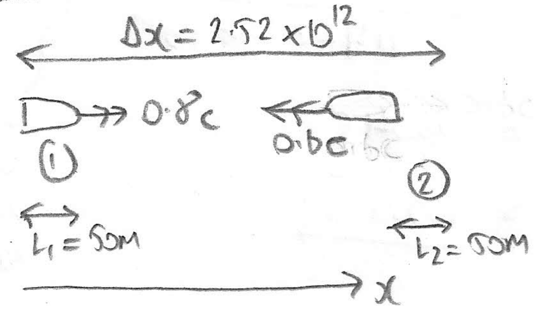
In relativistic limit $v \to c$, $\boxed{w \to v}$

$\left[\text{Since } v^2/c^2 \to 1, \; w \to v \left(\frac{3+1}{3+1} \right) = v \right]$

ie, $\boxed{w \to c}$

17/

Liz (on Earth)



(i) Proper length of rocket 1 is $(1 - 0.8^2)^{-\frac{1}{2}} \times 50 = \boxed{83.3\,m}$
(ie γL_1)

Proper length of rocket 2 is $(1 - 0.6^2)^{-\frac{1}{2}} \times 50 = \boxed{62.5\,m}$

(ii) If we take a reference frame where rocket ① is stationary, the observed velocity of rocket ② is

$$u_x' = \frac{u_x - V}{1 - \frac{u_x V}{c^2}} \qquad u_x = -0.6c \quad V = 0.8c$$

ie $$u_x' = \frac{-0.6c - 0.8c}{1 + 0.6 \times 0.8} = \boxed{-0.946c}$$

∴ rocket 2 will have length $\dfrac{62.5}{(1 - 0.946^2)^{-\frac{1}{2}}} = \boxed{20.3\,m}$

If we take a reference frame where rocket ② is stationary, the observed velocity of rocket ① is [using $V = -0.6c$, $u_x = 0.8c$]

$$u_x' = \frac{0.8c + 0.6c}{1 + 0.8 \times 0.6} = \boxed{0.946c}$$

∴ rocket 1 will have length $\dfrac{83.3}{(1 - 0.946^2)^{-\frac{1}{2}}} = \boxed{27.0 \text{ metres}}$

(iii) From Liz's perspective, relative velocity is $1.4c$ ← Since they are approaching each other

$\therefore$ time to collision is $t = \dfrac{2.52 \times 10^{12}}{1.4 \times 2.998 \times 10^{8}}$

$= 6004 \text{ s}$

$\approx \boxed{100 \text{ mins}}$

(iv) From Rocket 1's perspective, 100 mins in the Earth frame is $\dfrac{100}{(1-0.8^2)^{-\frac{1}{2}}} = \boxed{60 \text{ mins}}$

From Rocket 2's perspective, 100 mins in the Earth frame is $\dfrac{100}{(1-0.6^2)^{-\frac{1}{2}}} = \boxed{80 \text{ mins}}$

(v) If crew evacuation time is 50 min (in the rest frame of the rocket) this means an evacuation should be possible before the rockets collide.

June 2015