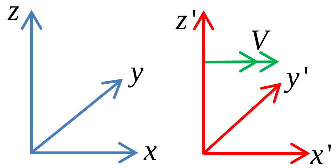
Special Relativity is a theory of dynamics proposed by **Albert Einstein** in 1905. The key mathematical element is the use of the ***Lorentz Transform*.** This extends the equations of *Galilean Relativity*, which relate the Cartesian x,y,z coordinates of an object to coordinates of the same object as viewed in a frame of reference moving at velocity V in the positive x direction relative to the x,y,z system. Let S denote the x,y,z coordinate system and S' denote the x',y',z' coordinates of the moving frame. The Lorentz transform incorporates the strange (but seeming true!) fact that the speed of light is the **same** for both S and S' frames. In other words, if a torch is shone from frame S, the speed of the light observed by S' would be the same speed as in S, and *not* the speed of light minus V. This is because **Maxwell's Equations**, which describe electric and magnetic fields, predict that electromagnetic waves propagate at a constant speed, independent of the (relative) velocity of any coordinate system. Einstein believed Maxwell's result to be the more fundamental (i.e. 'axiomatic') truth. This was helped by the experimental result of Michelson & Morely in 1887 which showed that there was no 'luminiferous aether' that light moved through. Light can propagate perfectly well through empty space (a vacuum). The consequences of Special Relativity are *profound*. It results in *length contraction*, *time dilation* and *time synchronisation* changes between the S and S' frames.

Galilean relativity

$$x = x'+Vt \qquad x' = x - Vt \qquad y = y' \qquad z = z' \qquad t = t'$$



Galilean relativity appears to work just fine in normal scenarios on Earth, i.e. when $V \ll c$ where the speed of light $c = 2.998\times 10^8\,\mathrm{ms}^{-1}$

The effects of Special relativity are *only significant* when V is close to c.

Consider the following candidate expressions for the Lorentz transform of the spatial coordinates between the S and S' frames:

$$x = \gamma(x'+Vt') \qquad x' = \gamma(x - Vt) \qquad y = y' \qquad z = z'$$

γ is a function of V. In order to be consistent with Galilean relativity, it must be *unity* when $V \ll c$

Note we have *not asserted* that time progresses at the same rate in each frame

Hence:

$$x = \gamma(x'+Vt')$$

$$\frac{x}{\gamma} = x'+Vt'$$

$$t' = \frac{x}{\gamma V} - \frac{x'}{V}$$

$$t' = \frac{x}{\gamma V} - \frac{\gamma(x - Vt)}{V}$$

$$\therefore t' = \gamma\left(t - \frac{x}{V}\left(1 - \frac{1}{\gamma^2}\right)\right)$$

$$x' = \gamma(x - Vt)$$

$$\frac{x'}{\gamma} = x - Vt$$

$$t = \frac{x}{V} - \frac{x'}{\gamma V}$$

$$t = \frac{\gamma(x'+Vt')}{V} - \frac{x'}{\gamma V}$$

$$\therefore t = \gamma\left(t' + \frac{x'}{V}\left(1 - \frac{1}{\gamma^2}\right)\right)$$

Now consider a *spherical light pulse* emitted when the origins of S and S' coincide. Since it radiates out at speed c in ***both*** S and S' from their (respective) origins, we can compare the radii r, r' of the pulse as observed from S and S'

$$r'^2 = c^2t'^2 = x'^2 + y'^2 + z'^2$$

$$r^2 = c^2t^2 = x^2 + y^2 + z^2$$

Since $y = y', \; z = z'$ this means $c^2t'^2 - x'^2 = c^2t^2 - x^2$

Now when $x' = 0, \quad x = Vt$

Hence $c^2t'^2 = c^2t^2 - V^2t^2$

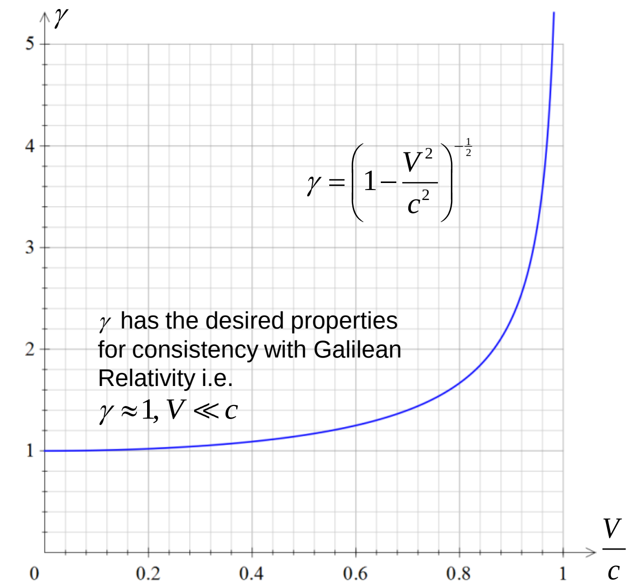
$$\Rightarrow t' = t\sqrt{1 - \frac{V^2}{c^2}}$$

Now using $t = \gamma\left(t' + \frac{x'}{V}\left(1 - \frac{1}{\gamma^2}\right)\right)$ when $x' = 0 \;\; \Rightarrow t = \gamma t\sqrt{1 - \frac{V^2}{c^2}}$

$$\therefore \gamma = \left(1 - \frac{V^2}{c^2}\right)^{-\frac{1}{2}}$$

$$\therefore 1 - \frac{1}{\gamma^2} = 1 - 1 - \frac{V^2}{c^2} = \frac{V^2}{c^2}$$

$$\therefore \frac{1}{V}\left(1 - \frac{1}{\gamma^2}\right) = \frac{V}{c^2}$$



The **Lorentz Transform** is now revealed!

$$x = \gamma(x'+Vt') \qquad x' = \gamma(x - Vt)$$

$$y = y' \qquad y = y'$$

$$z = z' \qquad z = z'$$

$$t = \gamma\left(t' + \frac{Vx'}{c^2}\right) \qquad t' = \gamma\left(t - \frac{Vx}{c^2}\right)$$

So ***lengths contract*** and ***time dilates and shifts*** when V becomes close to c

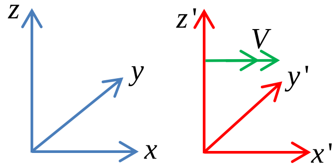
So time progresses *differently* in S and S' *unless* γ is unity.

Derivation of Lorentz transformations using the wave equation*

Galilean relativity

Galilean relativity appears to work just fine in normal scenarios on Earth, i.e. when $V \ll c$ where the speed of light $c = 2.998\times10^{8}\,\mathrm{ms}^{-1}$
The effects of Special relativity are *only significant* when V is close to c.

$$x = x' + Vt$$
$$x' = x - Vt$$
$$y = y'$$
$$z = z'$$
$$t = t'$$



Consider the following candidate expressions for the Lorentz transform of the spatial coordinates between the S and S' frames:

$$x = \gamma\left(x' + Vt'\right)$$
$$x' = \gamma\left(x - Vt\right)$$
$$y = y'$$
$$z = z'$$

γ is a function of V. In order to be consistent with Galilean relativity, it must be *unity* when $V \ll c$

Note we have *not asserted* that time progresses at the same rate in each frame

Hence:

$$x = \gamma\left(x' + Vt'\right) \qquad x' = \gamma\left(x - Vt\right)$$

$$\frac{x}{\gamma} = x' + Vt' \qquad \frac{x'}{\gamma} = x - Vt$$

$$t' = \frac{x}{\gamma V} - \frac{x'}{V} \qquad t = \frac{x}{V} - \frac{x'}{\gamma V}$$

$$t' = \frac{x}{\gamma V} - \frac{\gamma\left(x - Vt\right)}{V} \qquad t = \frac{\gamma\left(x' + Vt'\right)}{V} - \frac{x'}{\gamma V}$$

$$\therefore t' = \gamma\left(t - \frac{x}{V}\left(1 - \frac{1}{\gamma^2}\right)\right) \qquad \therefore t = \gamma\left(t' + \frac{x'}{V}\left(1 - \frac{1}{\gamma^2}\right)\right)$$

Einstein's postulate states that light propagates at speed c in both S and S' frames. We can therefore write down the following *wave equations* for light waves propagating in these frames.

wave amplitude

$$\psi = \psi\left(x - ct\right), \quad \frac{\partial^2\psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

$$\psi = \psi\left(x' - ct'\right), \quad \frac{\partial^2\psi}{\partial x'^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t'^2}$$

Using the chain rule, and $x = \gamma\left(x' + Vt'\right)$ $\quad t' = \gamma\left(t - \frac{x}{V}\left(1 - \frac{1}{\gamma^2}\right)\right)$ $\quad x' = \gamma\left(x - Vt\right)$ $\quad t = \gamma\left(t' + \frac{x'}{V}\left(1 - \frac{1}{\gamma^2}\right)\right)$

$$\frac{\partial\psi}{\partial t'} = \frac{\partial\psi}{\partial t}\frac{\partial t}{\partial t'} + \frac{\partial\psi}{\partial x}\frac{\partial x}{\partial t'}$$

$$\frac{\partial\psi}{\partial t'} = \frac{\partial\psi}{\partial t}\gamma + \frac{\partial\psi}{\partial x}\gamma V$$

$$\frac{\partial^2\psi}{\partial t'^2} = \gamma\frac{\partial^2\psi}{\partial t^2}\frac{\partial t}{\partial t'} + \gamma\frac{\partial^2\psi}{\partial x\partial t}\frac{\partial x}{\partial t'} + \gamma V\frac{\partial^2\psi}{\partial x^2}\frac{\partial x}{\partial t'} + \gamma V\frac{\partial^2\psi}{\partial x\partial t}\frac{\partial t}{\partial t'}$$

$$\frac{\partial^2\psi}{\partial t'^2} = \gamma^2\frac{\partial^2\psi}{\partial t^2} + +\gamma^2V^2\frac{\partial^2\psi}{\partial x^2} + 2\gamma^2V\frac{\partial^2\psi}{\partial x\partial t}$$

$$\frac{\partial\psi}{\partial x'} = \frac{\partial\psi}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial\psi}{\partial t}\frac{\partial t}{\partial x'}$$

$$\frac{\partial\psi}{\partial x'} = \frac{\partial\psi}{\partial x}\gamma + \frac{\partial\psi}{\partial t}\frac{1}{V}\left(\gamma - \frac{1}{\gamma}\right)$$

$$\frac{\partial^2\psi}{\partial x'^2} = \gamma\frac{\partial^2\psi}{\partial x^2}\frac{\partial x}{\partial x'} + \gamma\frac{\partial^2\psi}{\partial x\partial t}\frac{\partial t}{\partial x'} + \frac{\partial^2\psi}{\partial t^2}\frac{\partial t}{\partial x'}\frac{1}{V}\left(\gamma - \frac{1}{\gamma}\right) + \frac{\partial^2\psi}{\partial x\partial t}\frac{\partial x}{\partial x'}\frac{1}{V}\left(\gamma - \frac{1}{\gamma}\right)$$

$$\frac{\partial^2\psi}{\partial x'^2} = \gamma^2\frac{\partial^2\psi}{\partial x^2} + \gamma\frac{\partial^2\psi}{\partial x\partial t}\frac{1}{V}\left(\gamma - \frac{1}{\gamma}\right) + \frac{\partial^2\psi}{\partial t^2}\frac{1}{V^2}\left(\gamma - \frac{1}{\gamma}\right)^2 + \frac{\partial^2\psi}{\partial x\partial t}\gamma\frac{1}{V}\left(\gamma - \frac{1}{\gamma}\right)$$

$$\frac{\partial^2\psi}{\partial x'^2} = \gamma^2\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial t^2}\frac{1}{V^2}\left(\gamma - \frac{1}{\gamma}\right)^2 + 2\gamma\frac{\partial^2\psi}{\partial x\partial t}\frac{1}{V}\left(\gamma - \frac{1}{\gamma}\right)$$

$$\frac{\partial^2\psi}{\partial x'^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t'^2}, \quad \frac{\partial^2\psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

$$\therefore \frac{\partial^2\psi}{\partial t^2}\frac{1}{V^2}\left(\gamma - \frac{1}{\gamma}\right)^2 + 2\gamma\frac{\partial^2\psi}{\partial x\partial t}\frac{1}{V}\left(\gamma - \frac{1}{\gamma}\right) = \frac{\gamma^2V^2}{c^2}\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \frac{2\gamma^2V}{c^2}\frac{\partial^2\psi}{\partial x\partial t}$$

$$\therefore \frac{1}{V^2}\left(\gamma - \frac{1}{\gamma}\right)^2 = \frac{\gamma^2V^2}{c^2}\frac{1}{c^2}$$ Coefficient of $\frac{\partial^2\psi}{\partial t^2}$

$$\therefore 2\gamma\frac{1}{V}\left(\gamma - \frac{1}{\gamma}\right) = \frac{2\gamma^2V}{c^2}$$ Coefficient of $\frac{\partial^2\psi}{\partial x\partial t}$

$$\therefore \left(1 - \frac{1}{\gamma^2}\right) = \frac{V^2}{c^2}$$ Coefficient of $\frac{\partial^2\psi}{\partial x\partial t}$

$$\therefore \gamma = \left(1 - \frac{V^2}{c^2}\right)^{-\frac{1}{2}}$$

Check for consistency:

$$\frac{1}{V^2}\left(\gamma - \frac{1}{\gamma}\right)^2 = \frac{\gamma^2V^2}{c^2}\frac{1}{c^2}$$ Coefficient of $\frac{\partial^2\psi}{\partial t^2}$

$$\frac{\gamma^2}{V^2}\left(1 - \frac{1}{\gamma^2}\right)^2 = \frac{\gamma^2V^2}{c^2}\frac{1}{c^2}$$

$$\left(1 - \frac{1}{\gamma^2}\right)^2 = \frac{V^4}{c^4}$$

$$\left(1 - \frac{1}{\gamma^2}\right) = \frac{V^2}{c^2} \quad \therefore \gamma = \left(1 - \frac{V^2}{c^2}\right)^{-\frac{1}{2}}$$

The **Lorentz Transform** is now revealed!

$$x = \gamma\left(x' + Vt'\right) \qquad x' = \gamma\left(x - Vt\right)$$
$$y = y' \qquad y = y'$$
$$z = z' \qquad z = z'$$
$$t = \gamma\left(t' + \frac{Vx'}{c^2}\right) \qquad t' = \gamma\left(t - \frac{Vx}{c^2}\right)$$

So ***lengths contract** and **time dilates and shifts*** when V becomes close to c

*Idea from John Cullerne, Winchester College 2017.

The **Lorentz Transform** can be applied to relate **other dynamical parameters** between the S and S' frames

Velocity

$$v_x = \frac{dx}{dt} = \frac{\gamma(dx' + Vdt')}{\gamma\left(dt' + \frac{V}{c^2}dx'\right)}$$

$$v_x = \frac{\frac{dx'}{dt'} + V}{1 + \frac{V}{c^2}\frac{dx'}{dt'}}$$

$$\boxed{v_x = \frac{v'_x + V}{1 + \frac{v'_x V}{c^2}}}$$

$$v_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + \frac{V}{c^2}dx'\right)}$$

$$v_y = \frac{\frac{dy'}{dt'}}{\gamma\left(1 + \frac{V}{c^2}\frac{dx'}{dt'}\right)}$$

$$\boxed{v_y = \frac{v'_y}{\gamma\left(1 + \frac{v'_x V}{c^2}\right)}}$$

$$v_z = \frac{dz}{dt} = \frac{dz'}{\gamma\left(dt' + \frac{V}{c^2}dx'\right)}$$

$$v_z = \frac{\frac{dz'}{dt'}}{\gamma\left(1 + \frac{V}{c^2}\frac{dx'}{dt'}\right)}$$

$$\boxed{v_z = \frac{v'_z}{\gamma\left(1 + \frac{v'_z V}{c^2}\right)}}$$

and by an equivalent argument

$$\boxed{v'_x = \frac{v_x - V}{1 - \frac{v_x V}{c^2}} \qquad v'_y = \frac{v_y}{\gamma\left(1 - \frac{v_x V}{c^2}\right)} \qquad v'_z = \frac{v_z}{\gamma\left(1 - \frac{v_x V}{c^2}\right)}}$$

If the velocity of a *photon* of light is defined in plane polars

$$v_x = c\cos\theta$$

$$v'_x = c\cos\theta'$$

$$\cos\theta = \frac{\cos\theta' + \frac{V}{c}}{1 + \frac{V}{c}\cos\theta'}$$

$$\cos\theta' = \frac{\cos\theta - \frac{V}{c}}{1 - \frac{V}{c}\cos\theta}$$

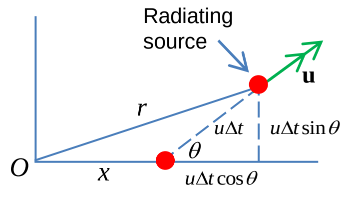
This is called '**relativistic aberration**'

Doppler shift

Consider a receding wave source of frequency f' in the S' frame. It crosses the x axis of the S frame at angle θ. and speed u. The velocity of waves emitted is w, in S.
The period T of waves received by an observer (in the x direction) at the origin O of the S frame is:

$$T = \Delta t + \frac{r - x}{w}$$

Δt: time between wave crests at *source*; w: wave speed; $r - x$: extra distance travelled by source between wave crests



From geometry:

$$r = \sqrt{(x + u\Delta t\cos\theta)^2 + u^2\Delta t^2\sin^2\theta}$$

$$r = \sqrt{x^2 + u^2\Delta t^2\cos^2\theta + 2ux\Delta t\cos\theta + u^2\Delta t^2\sin^2\theta}$$

$$r = \sqrt{x^2 + u^2\Delta t^2 + 2ux\Delta t\cos\theta}$$

$$r = x\sqrt{1 + 2\cos\theta\frac{u\Delta t}{x} + \left(\frac{u\Delta t}{x}\right)^2}$$

If $u\Delta t \ll x$ $\quad r \approx x\sqrt{1 + 2\cos\theta\frac{u\Delta t}{x}} \approx x\left(1 + \cos\theta\frac{u\Delta t}{x}\right) = x + u\Delta t\cos\theta$

$$\therefore \boxed{r - x \approx u\Delta t\cos\theta}$$

Hence frequency of radiation received at O is $f = 1/T$ where:

$$\frac{1}{f} = \Delta t + \frac{u\Delta t\cos\theta}{w} = \Delta t\left(1 + \frac{u\cos\theta}{w}\right)$$

$$\gamma = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}}$$

Using the **generalized Lorentz Transform** $\quad \Delta t = \gamma\left(\Delta t' + \frac{\mathbf{u}\cdot\Delta\mathbf{r}'}{c^2}\right)$

Now since the source is stationary in the S' frame $\quad \Delta\mathbf{r}' = 0$

Therefore $\Delta t = \gamma\Delta t' = \frac{\gamma}{f'}$

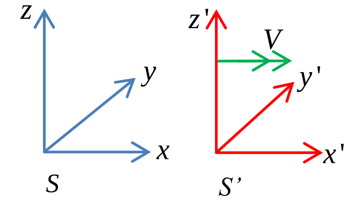
$$\frac{1}{f} = \frac{\gamma}{f'}\left(1 + \frac{u\cos\theta}{w}\right) \Rightarrow$$

Hence:

$$\boxed{f = \frac{f'}{\gamma\left(1 + \frac{u\cos\theta}{w}\right)}}$$

The Lorentz Transform

$$\boxed{\begin{aligned} x &= \gamma(x' + Vt') & x' &= \gamma(x - Vt) \\ y &= y' & y &= y' \\ z &= z' & z &= z' \\ t &= \gamma\left(t' + \frac{Vx'}{c^2}\right) & t' &= \gamma\left(t - \frac{Vx}{c^2}\right) \end{aligned}}$$



We can generalize to an S' velocity which is not parallel to the x axis of the S frame

$$\mathbf{r} = (x, y, z), \quad \mathbf{r}' = (x', y', z')$$

$$\mathbf{r} = \mathbf{r}' + \left(\frac{\gamma - 1}{V^2}(\mathbf{V}\cdot\mathbf{r}') + \gamma t'\right)\mathbf{V}$$

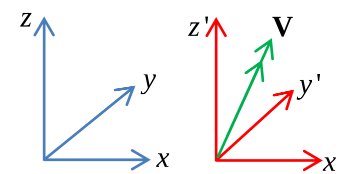
$$t = \gamma\left(t' + \frac{\mathbf{V}\cdot\mathbf{r}'}{c^2}\right)$$

$$\mathbf{r}' = \mathbf{r} + \left(\frac{\gamma - 1}{V^2}(\mathbf{V}\cdot\mathbf{r}) - \gamma t'\right)\mathbf{V}$$

$$t' = \gamma\left(t - \frac{\mathbf{V}\cdot\mathbf{r}}{c^2}\right)$$

$$V = |\mathbf{V}|$$

$$\gamma = \left(1 - \frac{V^2}{c^2}\right)^{-\frac{1}{2}}$$



Relativistic Doppler shift cont ....

$$f=\frac{f'}{\gamma\left(1+\frac{u\cos\theta}{w}\right)}$$

Define **Doppler frequency shift** $\Delta f = f - f'$

$$\Delta f=\frac{f'}{\gamma\left(1+\frac{u\cos\theta}{w}\right)}-f'$$

$$\frac{\Delta f}{f'}=\frac{1}{\gamma\left(1+\frac{u\cos\theta}{w}\right)}-1$$

Note if waves are *electromagnetic* $w=c$

The *classical* formula can easily be recovered by setting $\gamma = 1$

$$\frac{\Delta f}{f'}\approx\frac{1}{1+\frac{u\cos\theta}{w}}-1=\frac{1-1-\frac{u\cos\theta}{w}}{1+\frac{u\cos\theta}{w}}$$

$$\frac{\Delta f}{f'}\approx-\frac{\frac{u\cos\theta}{w}}{1+\frac{u\cos\theta}{w}}$$

If $u\cos\theta \ll w$

$$\frac{\Delta f}{f'}\approx-\frac{u\cos\theta}{w}$$

Unlike the classical formula, we get a *transverse Doppler effect* when $\theta = 90^\circ$ in the relativistic version

$$\frac{\Delta f}{f'}=\frac{1}{\gamma}-1$$

The Doppler shift is also related to the '***redshift***' z of a moving, radiating source

$$z=\frac{f'-f}{f}=\frac{f'}{f}-1$$

$$z=\gamma\left(1+\frac{u\cos\theta}{w}\right)-1$$

Momentum

We might expect 'force = rate of change of momentum' to be true in a relativistic sense as well as in the classical. However, the speed limit of c would imply an *upper limit on the amount of momentum a given mass could* have, if we use the classical momentum formula

$$\mathbf{p}=m\mathbf{u}$$

This would be *counter to reality* – we could easily devise a theoretical system which applies a finite amount of power, indefinitely, to a fixed mass system. e.g. a ball rolling down a infinitely long slope!
To get around this problem, let us *redefine* momentum such that it *can* become infinite as velocity tends towards c. i.e. multiply by γ....

$$\mathbf{p}=\gamma m\mathbf{u}$$

$$\gamma=\left(1-\frac{\mathbf{u}\cdot\mathbf{u}}{c^2}\right)^{-\frac{1}{2}}=\left(1-\frac{u^2}{c^2}\right)^{-\frac{1}{2}}$$

Some useful derivatives involving γ

$$\frac{d\gamma}{dt}=-\frac{1}{2}\left(1-\frac{\mathbf{u}\cdot\mathbf{u}}{c^2}\right)^{-\frac{3}{2}}\left(-2\frac{\mathbf{u}}{c^2}\cdot\frac{d\mathbf{u}}{dt}\right)$$

$$\frac{d\gamma}{du}=-\frac{1}{2}\left(1-\frac{u^2}{c^2}\right)^{-\frac{3}{2}}\left(-\frac{2u}{c^2}\right)$$

$$\frac{d\gamma}{dt}=\gamma^3\frac{\mathbf{a}\cdot\mathbf{u}}{c^2}$$

$\mathbf{a}=\frac{d\mathbf{u}}{dt}$ acceleration

$$\frac{d\gamma}{du}=\gamma^3\frac{u}{c^2}$$

Force, work & energy

$$1+\frac{\gamma^2u^2}{c^2}=\gamma^2$$

$$\Rightarrow\gamma^2\left(1-\frac{u^2}{c^2}\right)=1\Rightarrow\gamma=\left(1-\frac{u^2}{c^2}\right)^{-\frac{1}{2}}$$

$$\mathbf{f}=\frac{d}{dt}(\gamma m\mathbf{u})$$

$$\mathbf{f}=m\gamma\frac{d\mathbf{u}}{dt}+m\mathbf{u}\frac{d\gamma}{dt}$$

$$\mathbf{f}=m\gamma\mathbf{a}+m\gamma^3\left(\frac{\mathbf{a}\cdot\mathbf{u}}{c^2}\right)\mathbf{u}$$

'Relativistic Newton's Second Law'

$$W=\int\mathbf{f}\cdot d\mathbf{r}=\int\mathbf{f}\cdot\mathbf{u}dt$$

Work done

$$W=m\int\left(\gamma\mathbf{a}\cdot\mathbf{u}+\gamma^3\left(\frac{\mathbf{a}\cdot\mathbf{u}}{c^2}\right)u^2\right)dt$$

$$W=m\int\gamma(\mathbf{a}\cdot\mathbf{u})\left(1+\frac{\gamma^2u^2}{c^2}\right)dt$$

$$W=m\int\gamma^3(\mathbf{a}\cdot\mathbf{u})dt$$

$$1+\frac{\gamma^2u^2}{c^2}=\gamma^2$$

$$W=mc^2\int\gamma^3\frac{(\mathbf{a}\cdot\mathbf{u})}{c^2}dt$$

$$\frac{d\gamma}{dt}=\gamma^3\frac{\mathbf{a}\cdot\mathbf{u}}{c^2}$$

$$W=mc^2\int\frac{d\gamma}{dt}dt$$

$$W=mc^2\int_{\gamma_0}^{\gamma_1}d\gamma$$

$$W=(\gamma_1-\gamma_0)mc^2$$

So the **total energy** of a mass m is

$$E=\gamma mc^2$$

and when the *mass is at rest*

$$\gamma=1$$
$$E_0=mc^2$$

Hence **kinetic energy** is

$$E_k=(\gamma-1)mc^2$$

Now in **classical limit**

$u \ll u$

$$\gamma\approx1+\tfrac{1}{2}\frac{u^2}{c^2}$$

$$\therefore(\gamma-1)mc^2=\tfrac{1}{2}mu^2$$

Energy, momentum invariant

Consider the following quantity:

$$k = E^2 - |\mathbf{p}|^2 c^2$$
$$k = (\gamma m c^2)^2 - (\gamma m \mathbf{u})\cdot(\gamma m \mathbf{u}) c^2$$
$$k = \gamma^2 m^2 c^4 - \gamma^2 m^2 u^2 c^2$$
$$k = m^2 c^4 \gamma^2 \left(1 - \frac{u^2}{c^2}\right)$$
$$k = m^2 c^4 \left(1 - \frac{u^2}{c^2}\right)^{-1}\left(1 - \frac{u^2}{c^2}\right)$$
$$k = m^2 c^4$$

This is clearly an invariant, *regardless* of the frame of reference.

$$\boxed{E^2 - |\mathbf{p}|^2 c^2 = m^2 c^4}$$

Application: "A particle with rest mass *am* strikes a stationary particle with rest mass *bm*. The *am* particle had kinetic energy kmc^2, and the result was an inelastic collision, with no total energy release. Find the rest mass M of the resulting particle in terms of m"

$$(\gamma - 1) amc^2 = kmc^2 \quad \text{Kinetic energy}$$
$$\therefore \gamma = 1 + \tfrac{k}{a} = \tfrac{a+k}{a}$$
$$\gamma = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}} \quad \therefore 1 - \frac{u^2}{c^2} = \gamma^{-2} = \left(\tfrac{a}{a+k}\right)^2$$
$$\therefore \boxed{u = \sqrt{1 - \left(\tfrac{a}{a+k}\right)^2}\, c = \alpha c} \quad \text{Initial particle speed}$$
$$\alpha = \sqrt{1 - \left(\tfrac{a}{a+k}\right)^2}$$

$$E^2 - |\mathbf{p}|^2 c^2 = M^2 c^4 \quad \text{Energy-momentum invariant}$$
$$E = kmc^2 + amc^2 + bmc^2 = (k+a+b)mc^2$$
$$p = \gamma amu = \tfrac{a+k}{a} am \sqrt{1 - \left(\tfrac{a}{a+k}\right)^2}\, c$$
$$\therefore E^2 - p^2c^2 = (k+a+b)^2 m^2 c^4 - (a+k)^2 m^2 \left(1 - \left(\tfrac{a}{a+k}\right)^2\right) c^4$$
$$\therefore E^2 - p^2c^2 = m^2c^4 \left\{(k+a+b)^2 - (a+k)^2 - a^2\right\}$$
$$\therefore M^2c^4 = m^2c^4 \left\{(k+a+b)^2 - 2ak - k^2\right\}$$
$$\therefore M = m\sqrt{(k+a+b)^2 - 2ak - k^2}$$

Using **conservation of momentum,** we can also find the *velocity* of the resulting particle

$$\gamma amu = \tfrac{a+k}{a} am \sqrt{1 - \left(\tfrac{a}{a+k}\right)^2}\, c = \left(1 - \tfrac{v^2}{c^2}\right)^{-\frac{1}{2}} Mv$$
$$\therefore (a+k)\sqrt{1 - \left(\tfrac{a}{a+k}\right)^2}\, c = \left(1 - \tfrac{v^2}{c^2}\right)^{-\frac{1}{2}} v \sqrt{(k+a+b)^2 - 2ak - k^2}$$
$$(a+k)^2 \left(1 - \left(\tfrac{a}{a+k}\right)^2\right) c^2 = \left(1 - \tfrac{v^2}{c^2}\right)^{-1} v^2 \left((k+a+b)^2 - 2ak - k^2\right)$$
$$\frac{(a+k)^2 - a^2}{(k+a+b)^2 - 2ak - k^2} c^2 = v^2 \frac{c^2}{c^2 - v^2}$$
$$\boxed{\beta = \frac{2ak + k^2}{(k+a+b)^2 - 2ak - k}}$$
$$\beta(c^2 - v^2) = v^2$$
$$\beta c^2 = (1+\beta)v^2$$
$$\therefore \boxed{v = \sqrt{\frac{\beta}{1+\beta}} \times c}$$

Energy loss – which should be *zero* since no energy release.

$$\boxed{\Delta E = (k+a+b)mc^2 - \left(1 - \tfrac{v^2}{c^2}\right)^{-\frac{1}{2}} Mc^2}$$

Classical result:

Mass conservation and momentum conservation:

$$M = (a+b)m$$
$$amu = Mv$$
$$\therefore \boxed{v = \frac{a}{a+b} u}$$

Energy loss – not NOT zero in this case:

$$\Delta E = \tfrac{1}{2} amu^2 - \tfrac{1}{2} Mv^2$$
$$\Delta E = \tfrac{1}{2} amu^2 - \tfrac{1}{2}(a+b)m\left(\frac{a}{a+b}\right)^2 u^2$$
$$\Delta E = \tfrac{1}{2} amu^2 - \tfrac{1}{2}\frac{a^2}{a+b} mu^2$$
$$\Delta E = \tfrac{1}{2} amu^2 \left(1 - \frac{a}{a+b}\right)$$
$$\Delta E = \tfrac{1}{2} amu^2 \left(\frac{a+b-a}{a+b}\right)$$
$$\boxed{\Delta E = \frac{\tfrac{1}{2} abmu^2}{a+b}}$$

Worked example:

$$a = 2, b = 3, k = 1, \quad \therefore \boxed{\gamma = \tfrac{3}{2}}$$
$$\alpha = \sqrt{1 - \left(\tfrac{a}{a+k}\right)^2} = \sqrt{1 - \left(\tfrac{2}{3}\right)^2} = \tfrac{\sqrt{5}}{3}$$
$$\therefore \boxed{u = \tfrac{\sqrt{5}}{3} c}$$
$$M = m\sqrt{(k+a+b)^2 - 2ak - k^2}$$
$$M = m\sqrt{(1+2+3)^2 - 2(2)(1) - 1} = \sqrt{31}m \approx 5.57m$$
$$\beta = \frac{2ak + k^2}{(k+a+b)^2 - 2ak - k} = \frac{2(2)(1) + 1}{(1+2+3)^2 - 2(2)(1) - 1} = \boxed{\tfrac{5}{31}}$$
$$\therefore v = \sqrt{\frac{\beta}{1+\beta}} \times c = \boxed{\tfrac{\sqrt{5}}{6} c} \approx 0.373c$$
$$\Delta E = (1+2+3)mc^2 - \left(1 - \tfrac{5}{36}\right)^{-\frac{1}{2}} \sqrt{31} mc^2$$
$$\Delta E = 6mc^2 - \left(\tfrac{31}{36}\right)^{-\frac{1}{2}} \sqrt{31} mc^2$$
$$\Delta E = 6mc^2 - \left(\tfrac{31}{36}\right)^{-\frac{1}{2}} \sqrt{31} mc^2$$
$$\Delta E = 6mc^2 - 6mc^2 = 0$$

Classical version:

$$u = \tfrac{\sqrt{5}}{3} c$$
$$M = 5m$$
$$v = \tfrac{2}{5} u = \tfrac{2}{5} \tfrac{\sqrt{5}}{3} c$$
$$v = \tfrac{2}{3\sqrt{5}} c \approx 0.298c$$
$$\Delta E = \tfrac{3}{5} m \tfrac{5}{9} c^2 = \tfrac{1}{3} mc^2$$

Variant of calculation with *no change of rest mass*, but instead energy release as per Classical result.

$$M=(a+b)m$$

Using **conservation of momentum**

$$\gamma amu=\tfrac{a+k}{a}am\sqrt{1-\left(\tfrac{a}{a+k}\right)^2}c=\left(1-\tfrac{v^2}{c^2}\right)^{-\frac{1}{2}}(a+b)mv$$

$$\therefore (a+k)\sqrt{1-\left(\tfrac{a}{a+k}\right)^2}c=\left(1-\tfrac{v^2}{c^2}\right)^{-\frac{1}{2}}(a+b)v$$

$$(a+k)^2\left(1-\left(\tfrac{a}{a+k}\right)^2\right)c^2=\left(1-\tfrac{v^2}{c^2}\right)^{-1}v^2(a+b)^2$$

$$\frac{(a+k)^2-a^2}{(a+b)^2}c^2=v^2\frac{c^2}{c^2-v^2}$$

$$\beta=\frac{2ak+k^2}{(a+b)^2}$$

$$\beta\left(c^2-v^2\right)=v^2$$

$$\beta c^2=(1+\beta)v^2$$

$$\therefore v=\sqrt{\frac{\beta}{1+\beta}}\times c$$

Energy loss:

$$\Delta E=(k+a+b)mc^2-\left(1-\tfrac{v^2}{c^2}\right)^{-\frac{1}{2}}(a+b)mc^2$$

$$\Delta E=(k+a+b)mc^2-\left(1-\tfrac{\beta}{1+\beta}\right)^{-\frac{1}{2}}(a+b)mc^2$$

$$\Delta E=(k+a+b)mc^2-\left(\tfrac{1}{1+\beta}\right)^{-\frac{1}{2}}(a+b)mc^2$$

$$\Delta E=(k+a+b)mc^2-(1+\beta)^{\frac{1}{2}}(a+b)mc^2$$

$$\Delta E=(k+a+b)mc^2-\left(\frac{(a+b)^2+2ak+k^2}{(a+b)^2}\right)^{\frac{1}{2}}(a+b)mc^2$$

$$\frac{\Delta E}{mc^2}=k+a+b-\sqrt{(a+b)^2+2ak+k^2}$$

Worked example:

$$a=2,b=3,k=1$$

$$\alpha=\sqrt{1-\left(\tfrac{a}{a+k}\right)^2}=\sqrt{1-\left(\tfrac{2}{3}\right)^2}=\tfrac{\sqrt{5}}{3}$$

$$\therefore u=\tfrac{\sqrt{5}}{3}c$$

$$\beta=\frac{2ak+k^2}{(a+b)^2}=\frac{2(2)(1)+1}{(2+3)^2}=\tfrac{5}{25}=\tfrac{1}{5}$$

$$\therefore v=\sqrt{\frac{\beta}{1+\beta}}\times c=\tfrac{1}{\sqrt{6}}c\approx 0.408c$$

$$\frac{\Delta E}{mc^2}=k+a+b-\sqrt{(a+b)^2+2ak+k^2}$$

$$\frac{\Delta E}{mc^2}=1+2+3-\sqrt{(2+3)^2+2\times2\times1+1}\approx 0.523$$

Classical version:

$$u=\tfrac{\sqrt{5}}{3}c$$

$$M=5m$$

$$v=\tfrac{2}{5}u=\tfrac{2}{5}\tfrac{\sqrt{5}}{3}c$$

$$v=\tfrac{2}{3\sqrt{5}}c\approx 0.298c$$

$$\Delta E=\tfrac{3}{5}m\tfrac{5}{9}c^2=\tfrac{1}{3}mc^2$$

This may sound like a strange result, the energy loss in the relativistic case is larger and *also* the velocity of M is larger than in the Classical scenario.

However, the relativistic speed means the initial KE is larger in the relativistic case.

Classical KE $\quad E_k=\tfrac{1}{2}(2m)u^2=m\left(\tfrac{\sqrt{5}}{3}c\right)^2=\tfrac{5}{9}mc^2\approx 0.56mc^2$

Relativistic KE $\quad E_k=\left(\tfrac{3}{2}-1\right)(2m)c^2=mc^2$

NOTE: In Classical mechanics we ignore any momentum associated with the energy loss. e.g. random motion of molecules with net zero momentum but net average KE. In the relativistic scenario here we have assumed the same. However in particle physics we consider single particles, so we might incorporate a release of a photon, which will have momentum that will need to be incorporated in the conservation of momentum calculation:

$$p_{photon}=\frac{h}{\lambda}$$