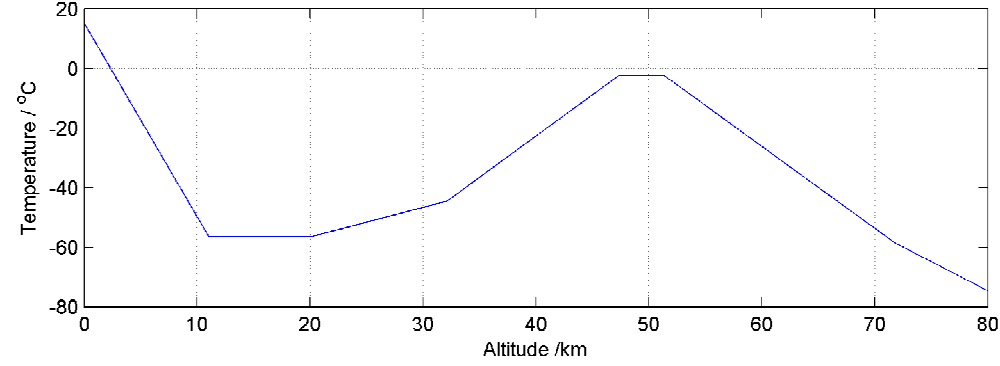
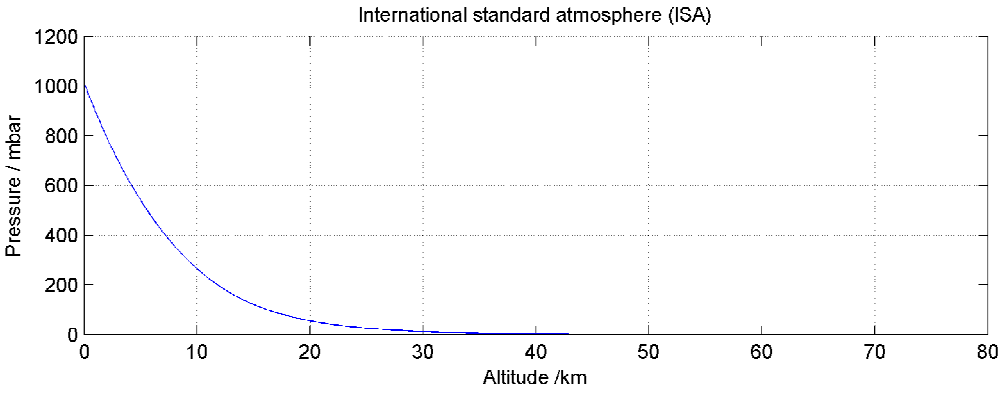
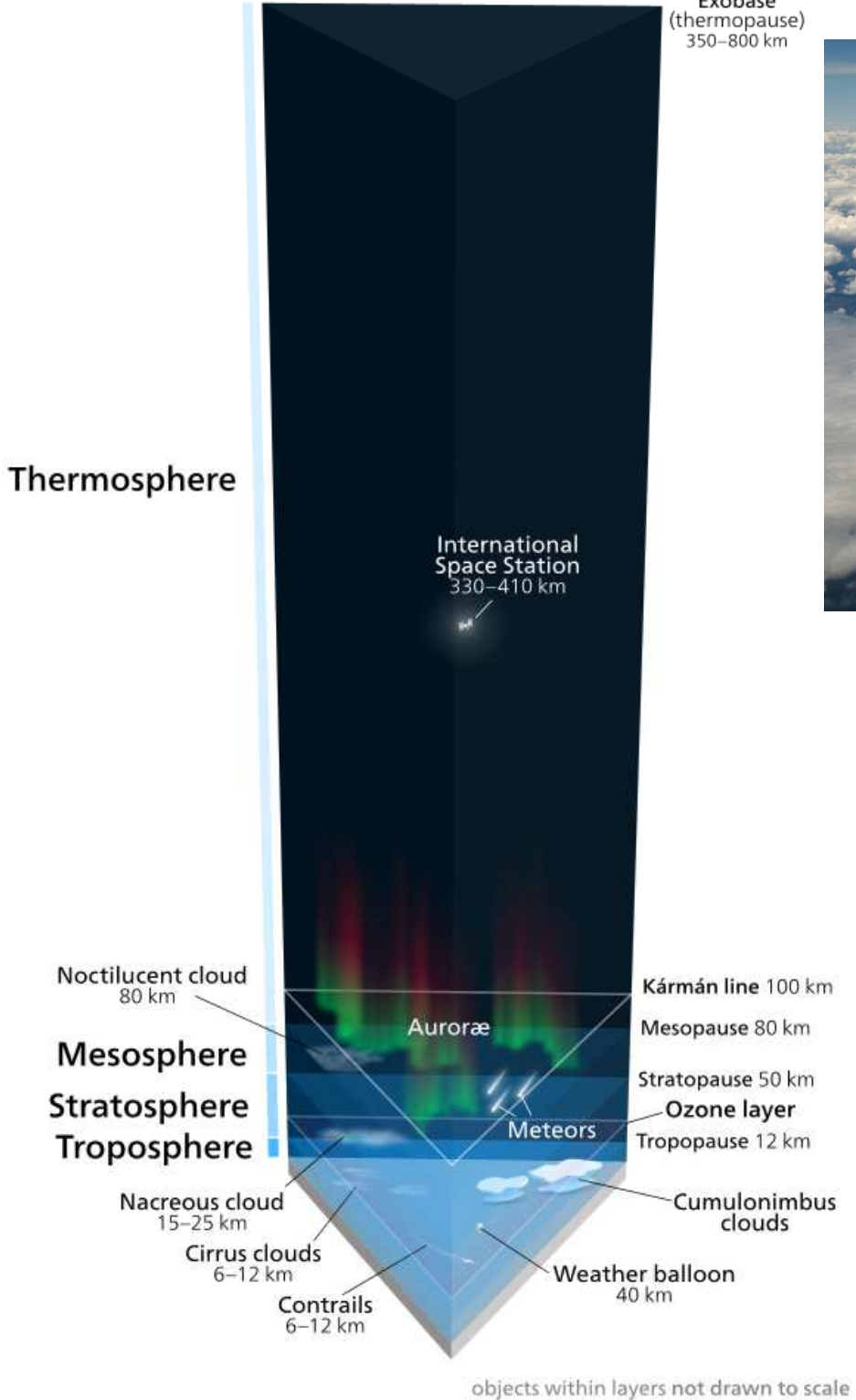


The Standard Atmosphere

Dr Andrew French

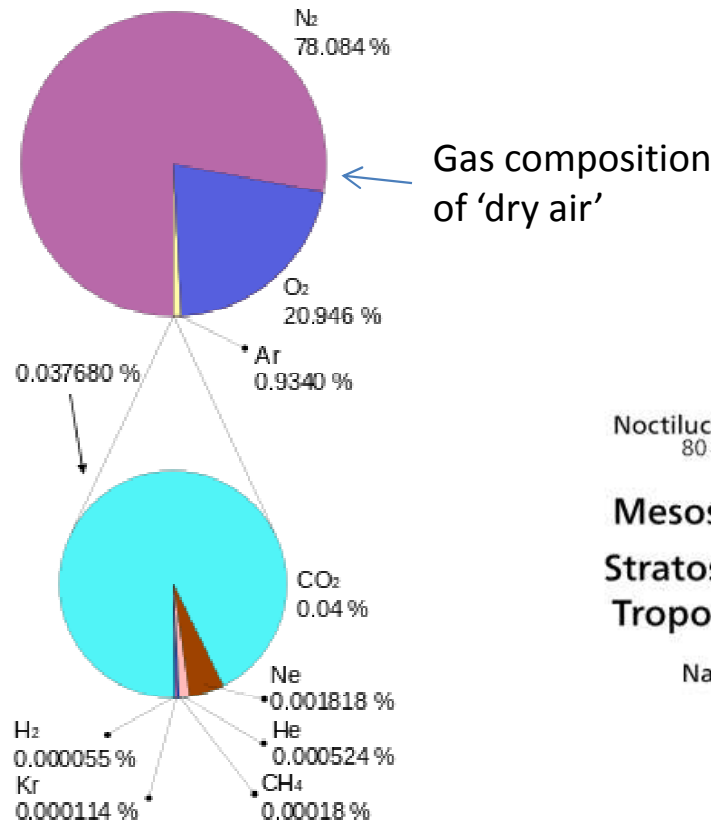


The International Standard Atmosphere (ISA) is an idealized model of the variation of average air pressure and temperature with altitude.

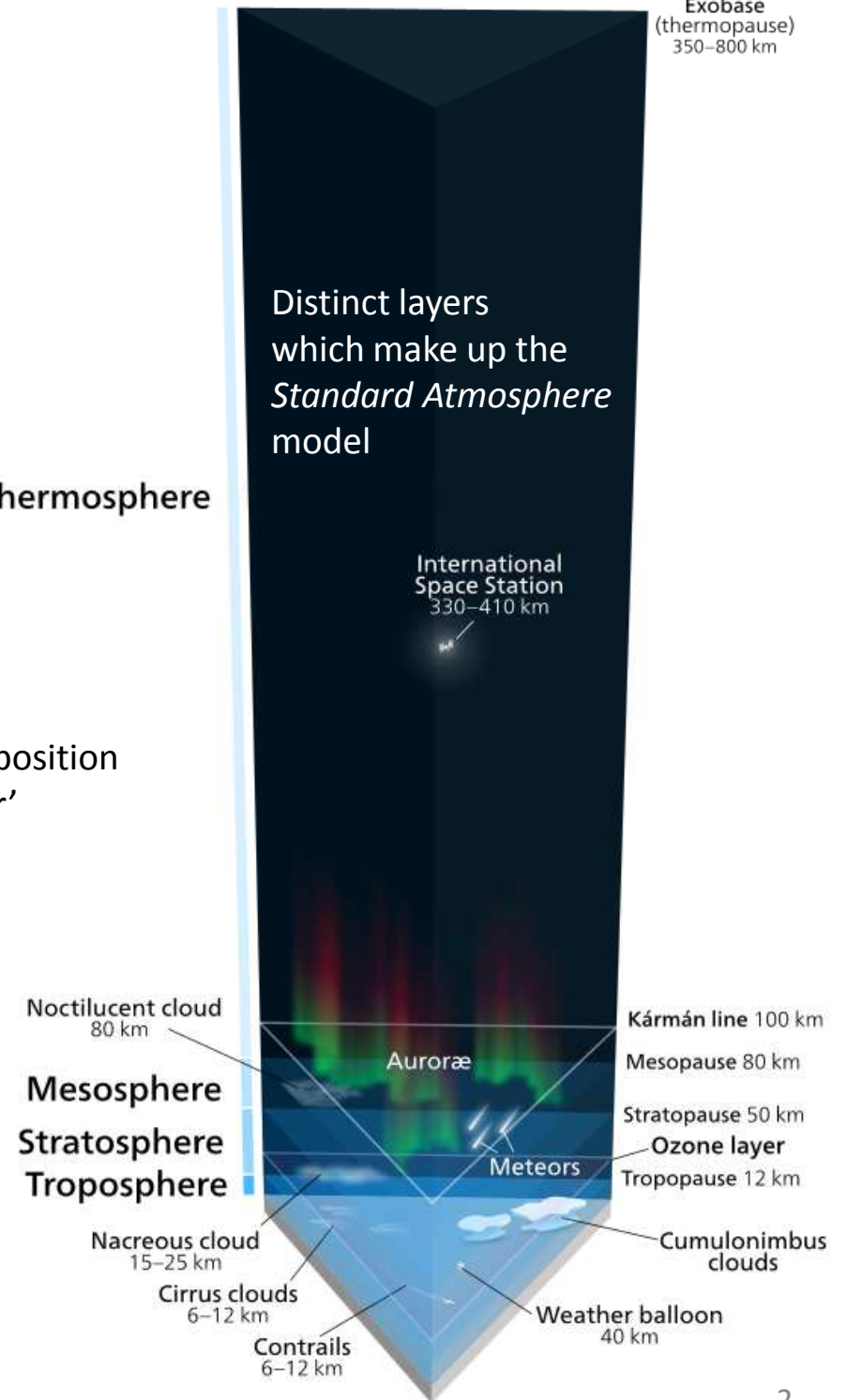
Assumptions:

- The atmosphere consists of a number of fixed layers, each with a constant temperature gradient or 'lapse rate'.

- The atmosphere consists of a *single ideal gas*, whose molecular mass takes into account the actual composition of different gases (e.g. Nitrogen, Oxygen, Carbon Dioxide, water vapour etc).

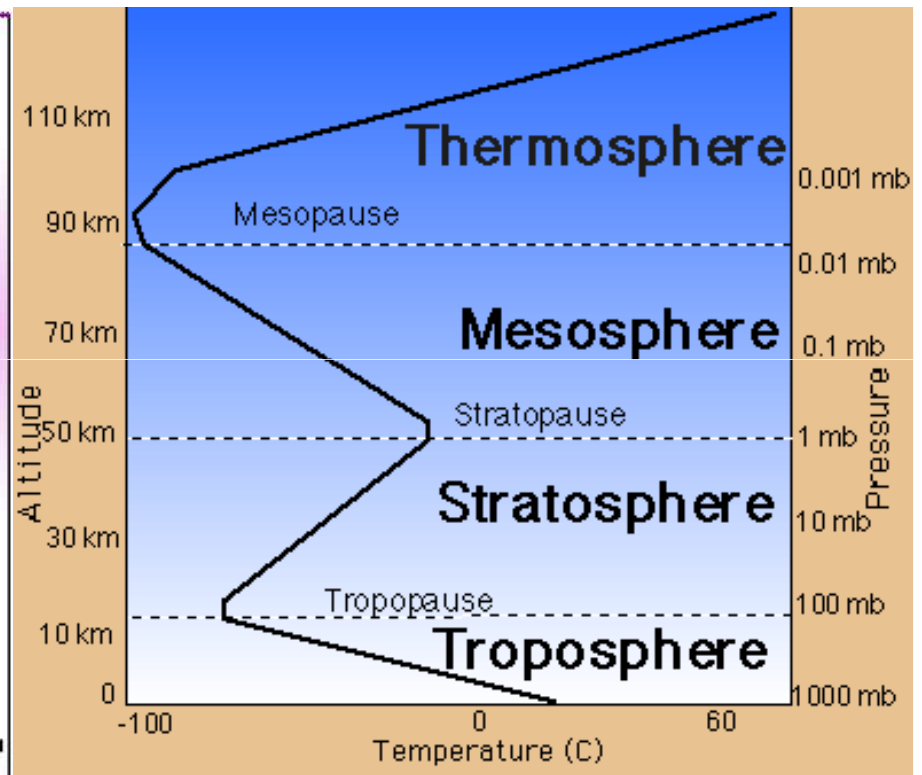
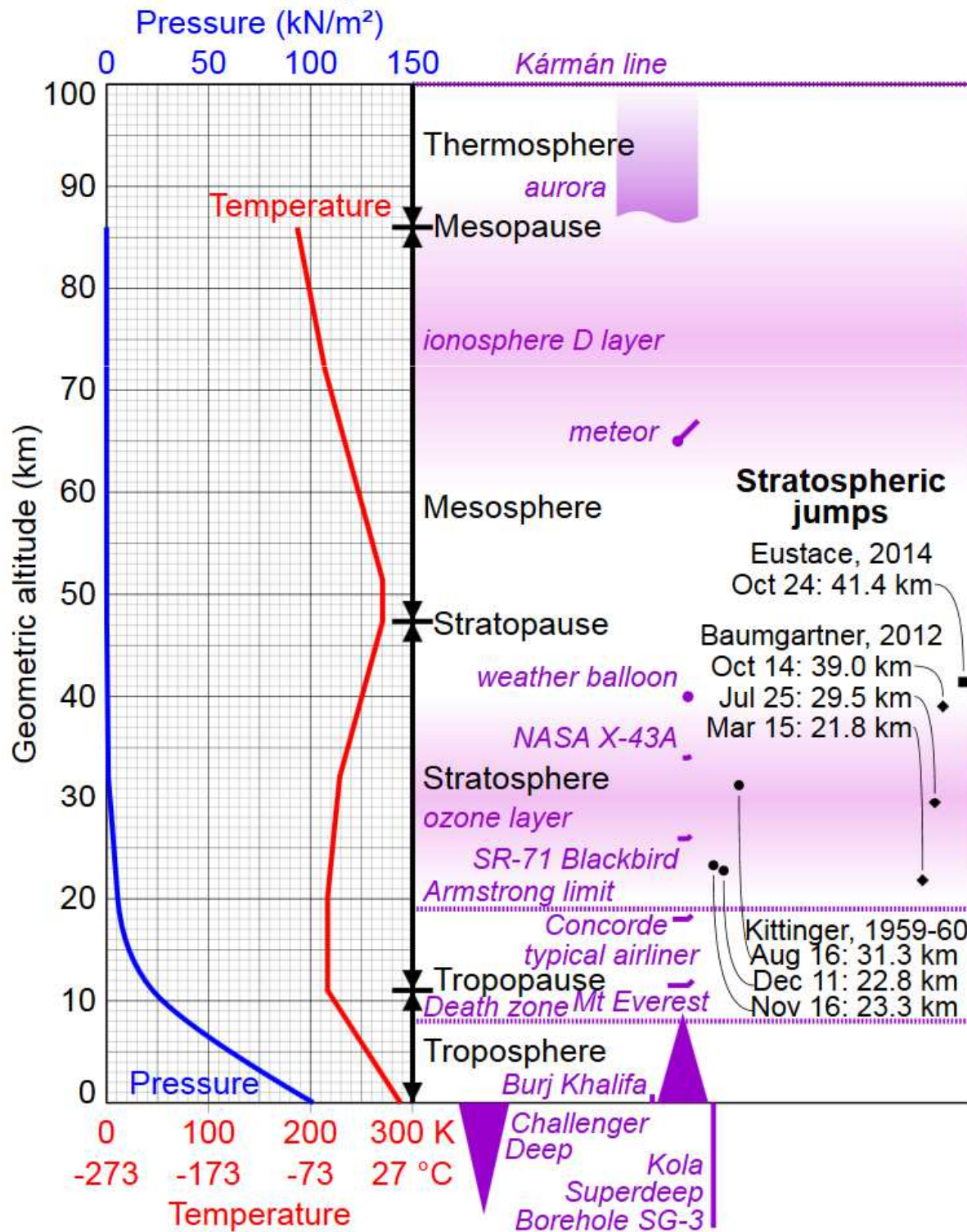


Thermosphere



International Standard Atmosphere

SpaceShipOne •

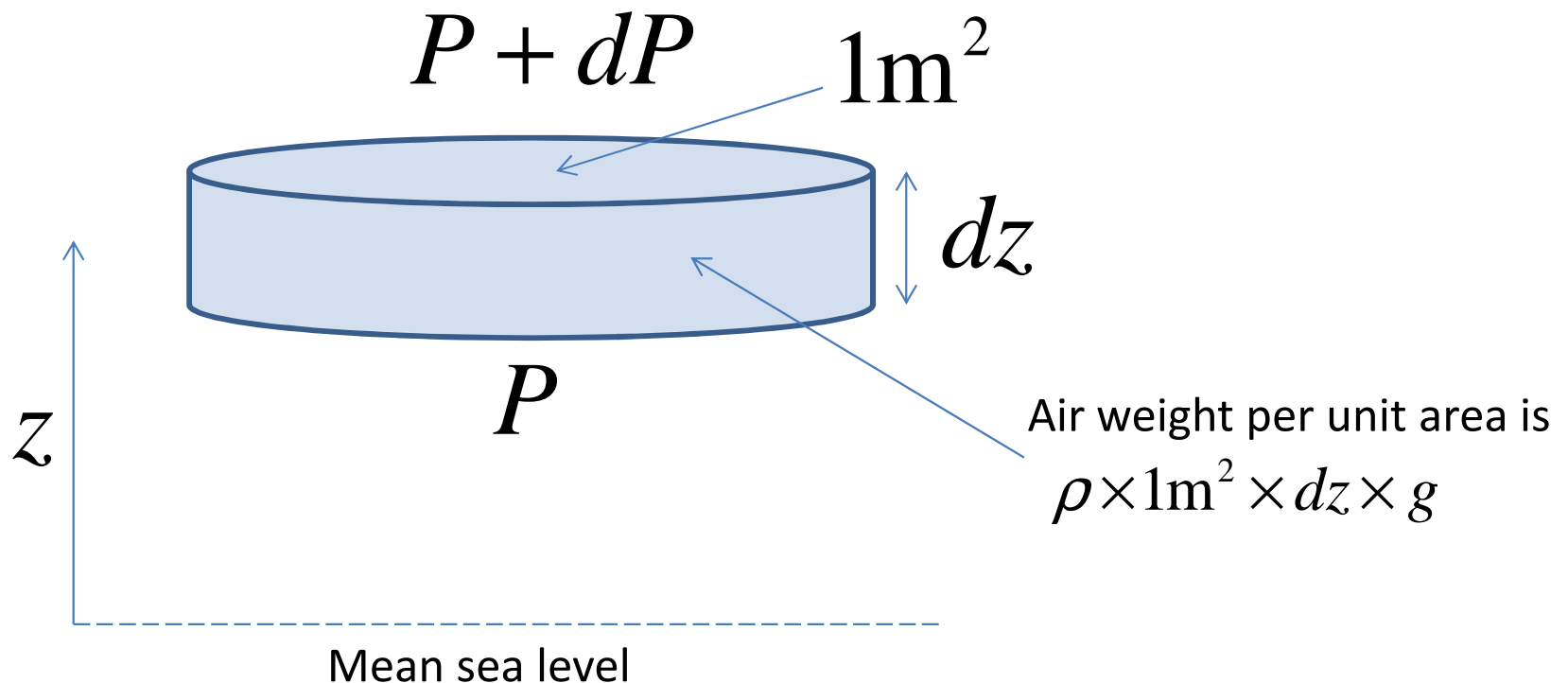


Note in the *thermosphere* the temperature (i.e. mean kinetic energy of air molecules) can rise to *several thousand degrees Celsius*. However, the air density is *extremely low*, which means it would still feel very cold indeed, as very little heat could be extracted from a given volume of air.

Determining air pressure

Consider a 1m^2 horizontal cross section parcel of air of density ρ at an altitude z , with vertical width dz . The atmospheric pressure change dP between altitudes z and $z+dz$ resulting from the removal of the air parcel from the total weight of air above is:

$$dP = -\rho g dz$$



Determining air pressure in the ISA

Let us assume that the air column is comprised of dry air with molar mass

$$M = 0.02896 \text{ kg mol}^{-1}$$

Let us assume that the air column is an ideal gas. If n moles of gas occupies volume V at pressure P (Pascals) and (absolute) temperature T (Kelvin):

$$PV = nRT \quad \text{Ideal gas equation} \quad \therefore \frac{P}{RT} = \frac{n}{V}$$

The density of the air is the mass of n moles divided by the volume

$$\rho = \frac{nM}{V}$$

Hence:

$$\rho = \frac{MP}{RT}$$

Change in weight of air column

$$dP = -\rho g dz$$

Ideal gas

$$\rho = \frac{MP}{RT}$$

$$\frac{dP}{P} = -\frac{M}{R} \frac{g dz}{T}$$

$$\int_{P_0}^P \frac{1}{P} dP = -\frac{M}{R} \int_{z_0}^z \frac{g}{T} dz$$

$$\ln\left(\frac{P}{P_0}\right) = -\frac{M}{R} \int_{z_0}^z \frac{g}{T} dz$$

To account for changes in gravitational field strength g with altitude, let us define a *geopotential altitude* h .

$$h = \frac{zgR_E}{g_0(z + R_E)}$$

$$R_E = 6378.1370 \text{ km}$$

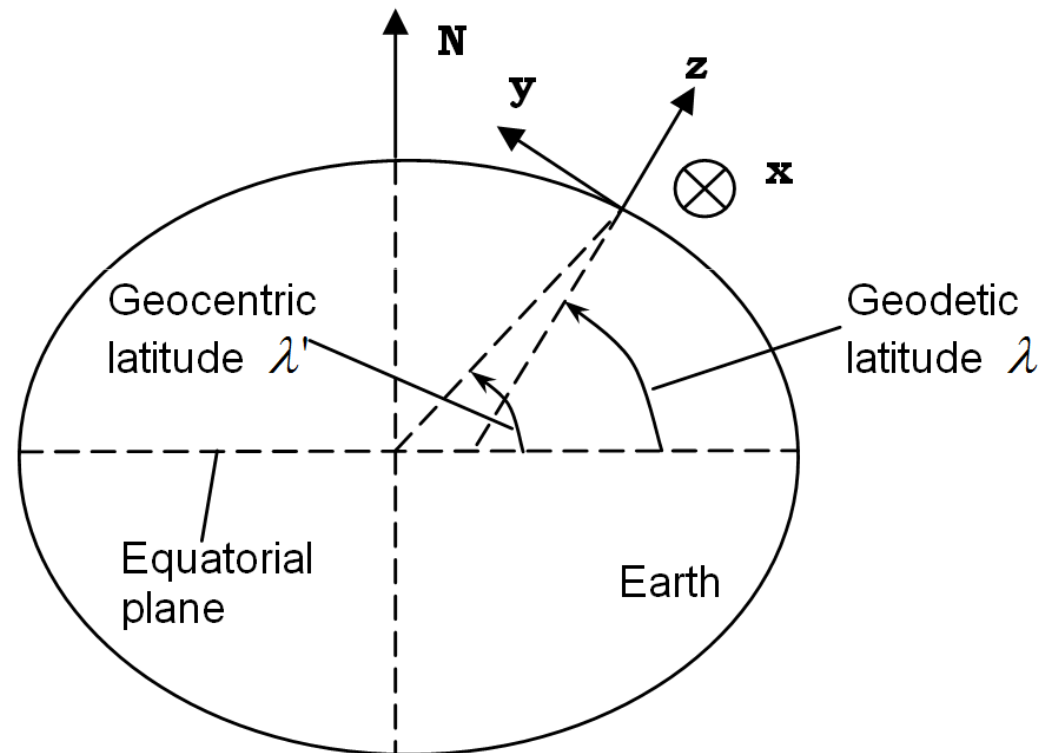
$$g = g_1 \left\{ 1 - a \cos(2\lambda) + b \cos^2(2\lambda) \right\}$$

$$g_0 = 9.80665 \text{ ms}^{-2}$$

$$g_1 = 9.80616 \text{ ms}^{-2}$$

$$a = 0.0026373$$

$$b = 0.0000059$$



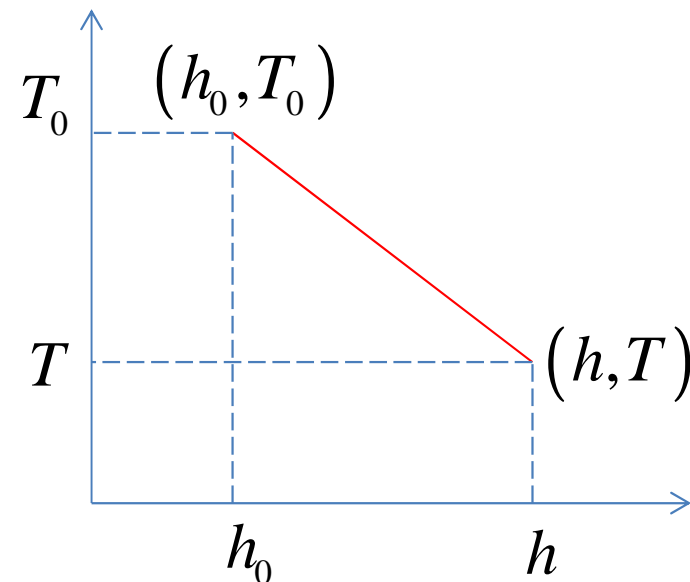
So g is now a function of latitude only, and our pressure integral is now in terms of geopotential height h

$$\ln\left(\frac{P}{P_0}\right) = -\frac{Mg}{R} \int_{h_0}^h \frac{1}{T} dh$$

Now let us define the temperature T (in Kelvin) to be a linear function of *geopotential* altitude. The (negative) gradient is defined to be the **lapse rate** L .

Note this means a negative lapse rate implies a rise in temperature with height.

$$T = T_0 - L(h - h_0)$$

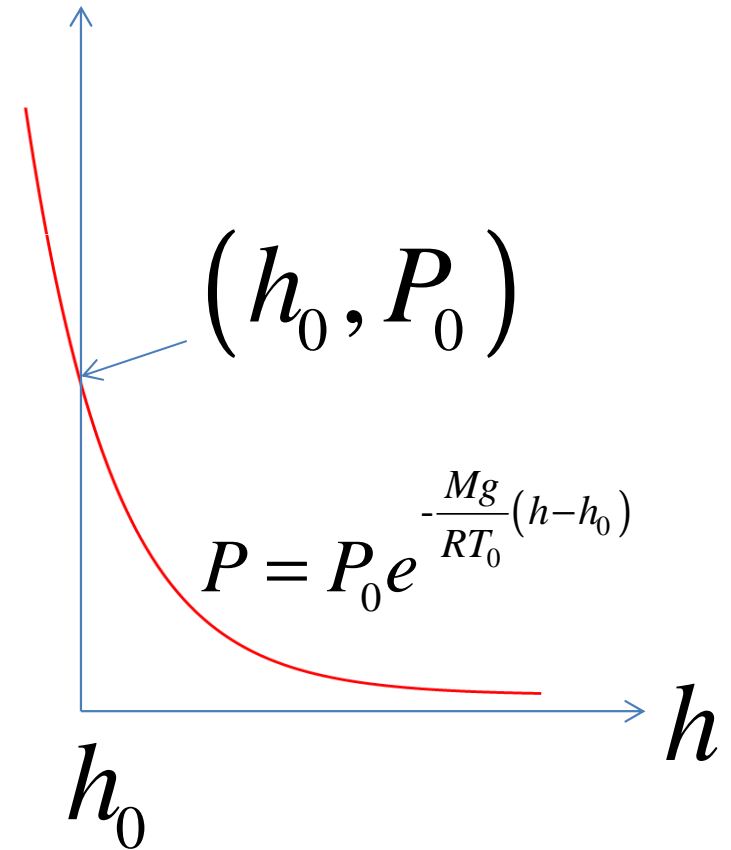


Case 1: Isothermal layer i.e. $L = 0$

$$T = T_0$$

$$\ln\left(\frac{P}{P_0}\right) = -\frac{Mg}{RT_0} \int_{h_0}^h dh$$

$$P = P_0 e^{-\frac{Mg}{RT_0}(h-h_0)}$$



Case 2: Constant but non-zero, lapse rate L

$$T = T_0 - L(h - h_0)$$

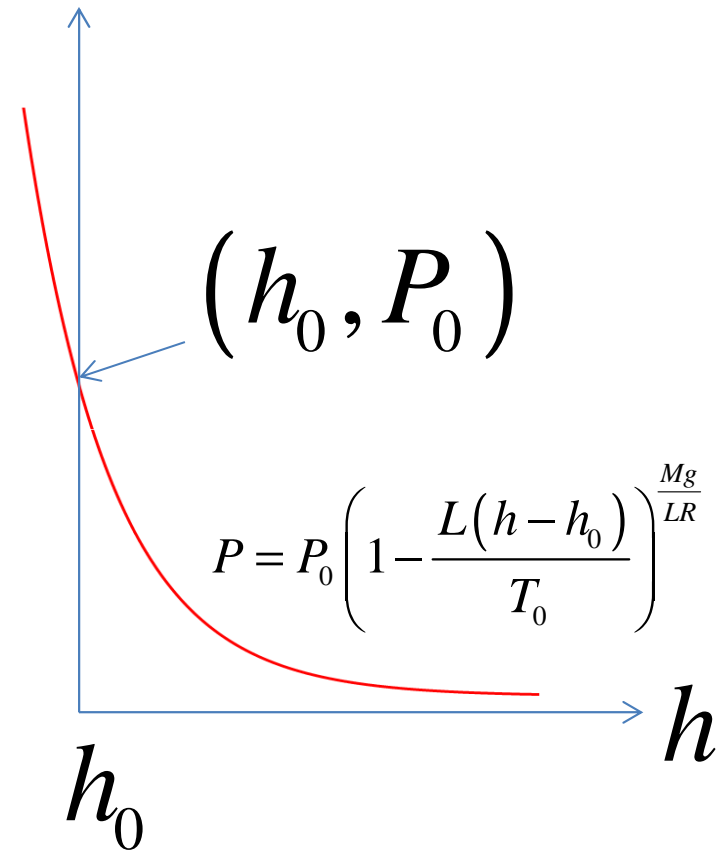
$$\ln\left(\frac{P}{P_0}\right) = -\frac{Mg}{R} \int_{h_0}^h \frac{1}{T_0 - L(h - h_0)} dh$$

$$\ln\left(\frac{P}{P_0}\right) = -\frac{Mg}{(-L)R} \int_{h_0}^h \frac{-L}{T_0 + Lh_0 - Lh} dh$$

$$\ln\left(\frac{P}{P_0}\right) = \frac{Mg}{LR} \left[\ln(T_0 + Lh_0 - Lh) \right]_{h_0}^h$$

$$\ln\left(\frac{P}{P_0}\right) = \frac{Mg}{LR} \ln\left(\frac{T_0 + Lh_0 - Lh}{T_0}\right)$$

$$\ln\left(\frac{P}{P_0}\right) = \ln\left(\left(\frac{T_0 + Lh_0 - Lh}{T_0}\right)^{\frac{Mg}{LR}}\right)$$



$$P = P_0 \left(1 - \frac{L(h - h_0)}{T_0}\right)^{\frac{Mg}{LR}}$$

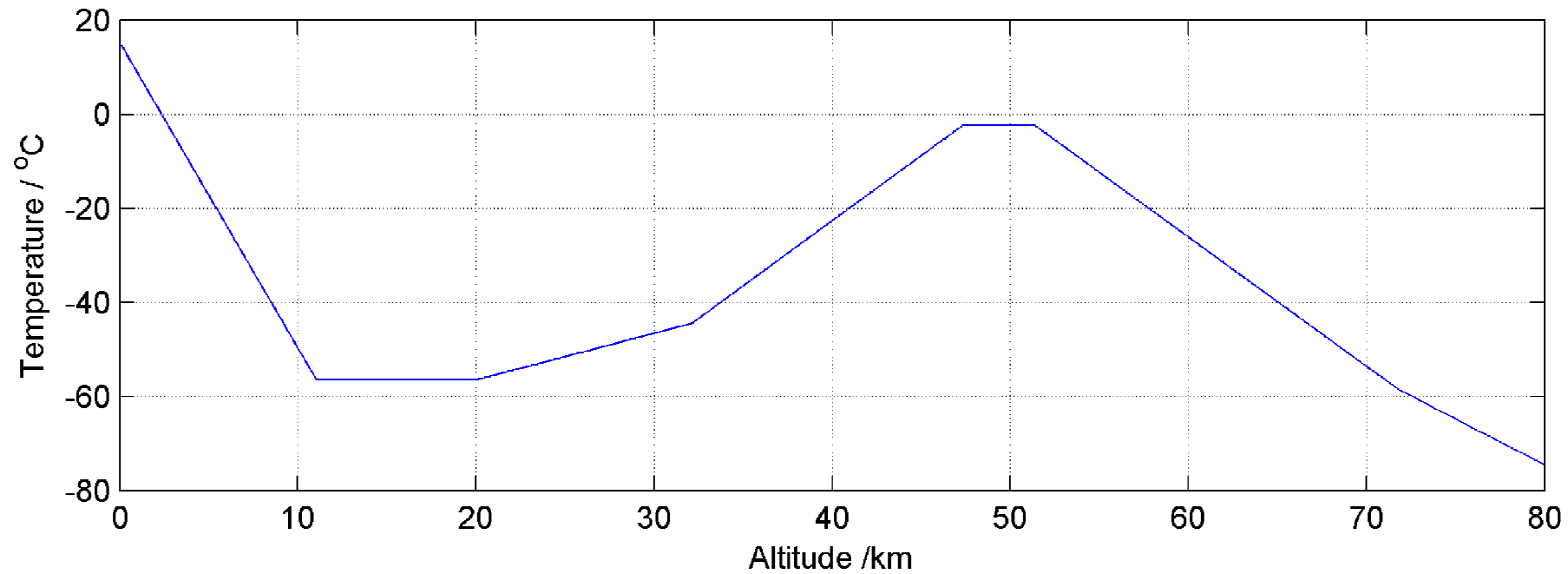
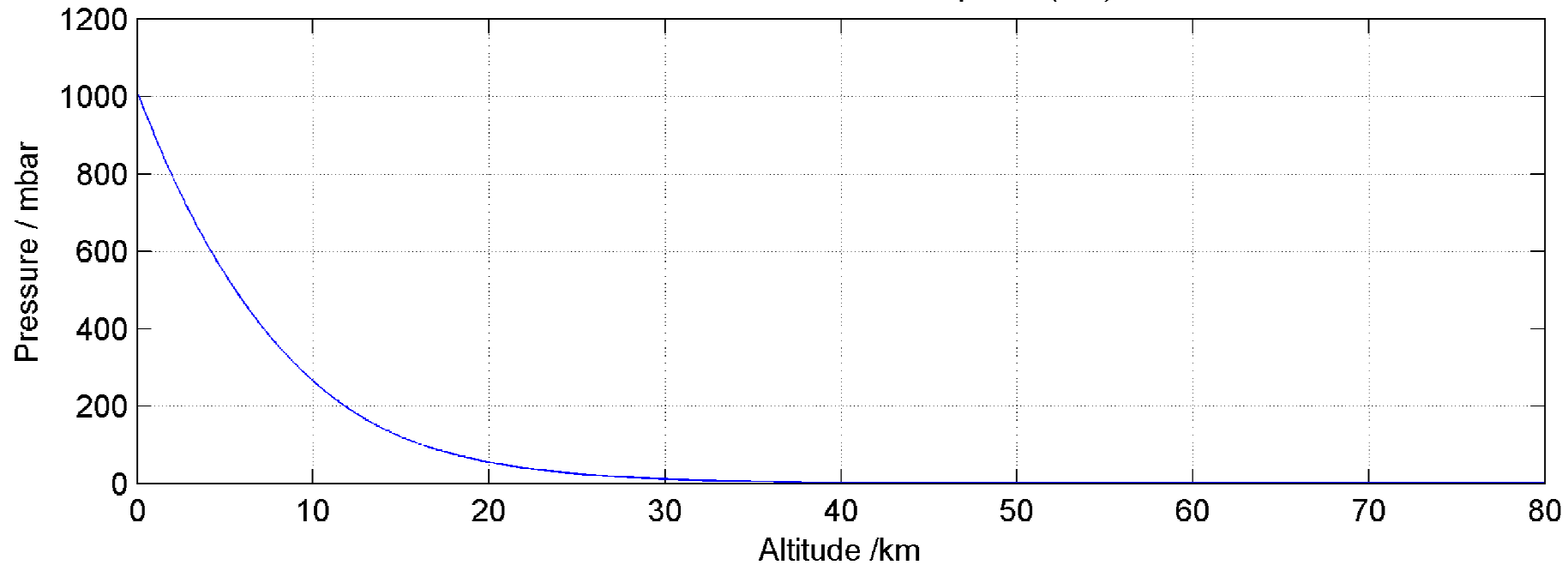
Layers in the ISA

Layer	Level Name	Base Geopotential Altitude above MSL ^[5] h (in km)	Base Geometric Altitude above MSL ^[5] z (in km)	Lapse Rate (in °C/km)	Base Temperature T (in °C)	Base Atmospheric Pressure p (in Pa)
0	Troposphere	0.0	0.0	-6.5	+15.0	101325
1	Tropopause	11.000	11.019	+0.0	-56.5	22632
2	Stratosphere	20.000	20.063	+1.0	-56.5	5474.9
3	Stratosphere	32.000	32.162	+2.8	-44.5	868.02
4	Stratopause	47.000	47.350	+0.0	-2.5	110.91
5	Mesosphere	51.000	51.413	-2.8	-2.5	66.939
6	Mesosphere	71.000	71.802	-2.0	-58.5	3.9564
7	Mesopause	84.852	86.000	—	-86.28	0.3734

In the above table, [geopotential height](#) is calculated from a mathematical model in which the acceleration due to gravity is assumed constant. Geometric height results from the assumption that gravity obeys an inverse square law.^[6]

The ISA model is based on average conditions at mid latitudes, as determined by the ISO's TC 20/SC 6 technical committee. It has been revised from time to time since the middle of the 20th century.

International standard atmosphere (ISA)



To generate the **Pressure and Temperature vs altitude** graphs on the previous slide:

1. State a desired range of altitude /km
2. Convert to *geopotential* altitude using

$$h = \frac{zgR_E}{g_0(z + R_E)}$$

$$g = g_1 \{1 - a \cos(2\lambda) + b \cos^2(2\lambda)\}$$

$$g_0 = 9.80665 \text{ms}^{-2}, \quad g_1 = 9.80616 \text{ms}^{-2}$$

$$a = 0.0026373, \quad b = 0.0000059$$

Note temperature T in degrees Celsius is:
 $273 + T$ Kelvin

3. Determine ISA layer base altitude and lapse rate
4. Determine Temperature and Pressure at the base of the layer (i.e. T_0 and P_0)
5. Compute pressure and temperature

$$T = T_0$$

$$P = P_0 e^{-\frac{Mg}{RT_0}(h-h_0)}$$

$$L = 0$$

$$T = T_0 - L(h - h_0)$$

$$P = P_0 \left(1 - \frac{L(h - h_0)}{T_0} \right)^{\frac{Mg}{LR}}$$

$$L \neq 0$$

i.e. the average pressure, temperature structure of the air column

What about humidity?

Any sensible description of non-arid climatology will need to take into account the impact of variable amounts of **water vapour** contained within the air column. In the *Troposphere* at least, the presence of water vapour can have a dramatic influence upon thermodynamic variables such as temperature and pressure, and is obviously a fundamental component of weather phenomena such as cloud and fog.

To model the effect of humidity upon temperature and pressure, let us modify our original single ideal gas assumption to consider a *composite* of dry air and water vapour. The molar masses of dry air and water vapour are, respectively:

$$M_d = 0.02896 \text{kgmol}^{-1}$$

$$M_v = 0.01802 \text{kgmol}^{-1}$$

The respective ideal gas equations are, for n_d moles of dry air and n_v moles of water vapour:

$$P = P_v + P_d$$

i.e. the overall pressure P is the sum of the 'partial pressures' of the component gases

$$P_v = \frac{n_v RT}{V} \quad P_d = \frac{n_d RT}{V}$$

Let us define *relative humidity* U as the ratio of water vapour pressure to that at saturation E_s

$$P_V = UE_s$$

An empirical model for saturation pressure as a pure function of temperature is given by the *Arden Buck equation*

http://en.wikipedia.org/wiki/Arden_Buck_equation

$$E_s = 6.1121e^{\left\{ \left(18.678 - \frac{T}{234.5} \right) \left(\frac{T}{T+257.14} \right) \right\}}$$

There are in fact many empirical formulae for vapour pressure.

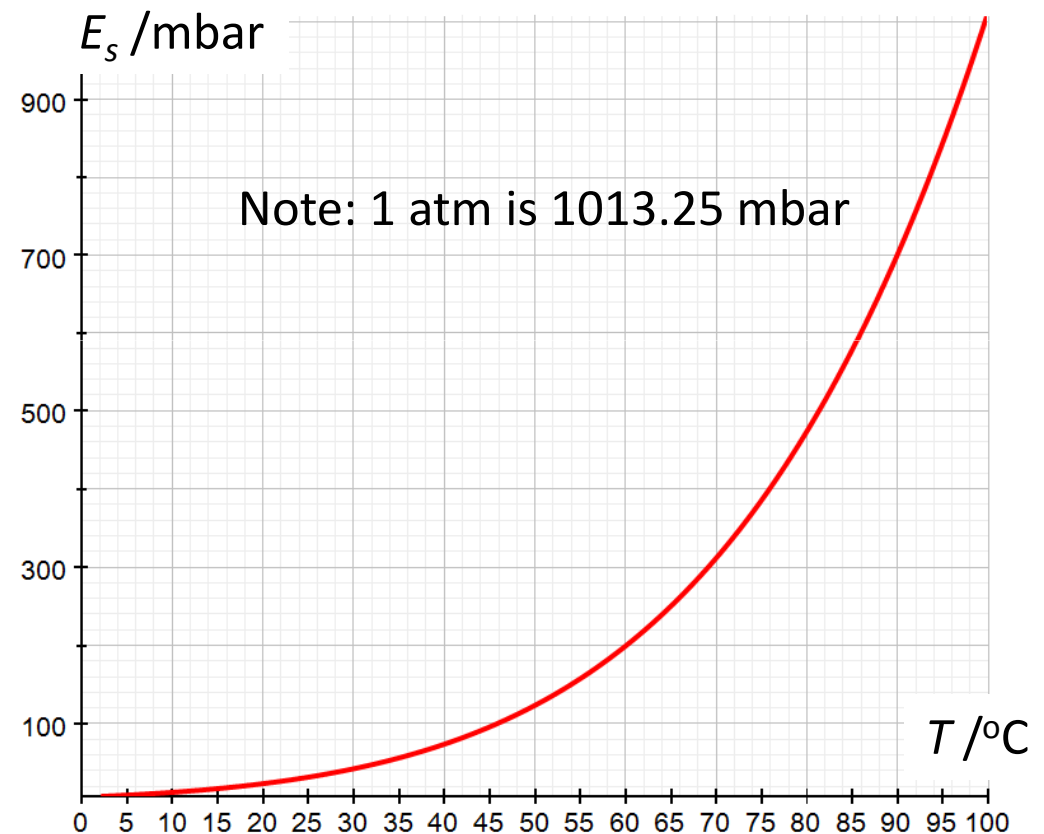
A good selection are compared at

<http://cires.colorado.edu/~voemel/vp.html>

where the saturation vapour pressure E_s is given in mbar and temperature in degrees Celsius

Note the vapour pressure is *very small* compared to the pressure of dry air when temperatures are low. Therefore to a very good approximation we can ignore it in the pressure calculation.

However, the lapse rate *is significantly* affected by humidity.....



The overall density of the atmosphere is:

$$\rho = \frac{n_d M_d + n_v M_v}{V} = \frac{M_d}{V} \left(n_d + n_v \frac{M_v}{M_d} \right)$$

From the ideal gas equations:

$$n_v = \frac{V}{RT} P_v = \frac{V}{RT} U E_s$$

$$n_d = \frac{V}{RT} P_d = \frac{V}{RT} (P - P_v) = \frac{V}{RT} (P - U E_s)$$

Hence:

$$\rho = \frac{M_d}{RT} \left(P - U \left(1 - \frac{M_v}{M_d} \right) E_s(T) \right)$$

The pressure integral is now more complex, but can be evaluated using a numerical method:

$$dP = -\rho g dh \quad \rho = \frac{M_d}{RT} \left(P - U \left(1 - \frac{M_v}{M_d} \right) E_s(T) \right) \quad T = T_0 - L(h - h_0)$$

$$\longrightarrow \frac{dP}{dh} = -\frac{M_d g}{RT} \left(P - U \left(1 - \frac{M_v}{M_d} \right) E_s(T) \right)$$

A simple iterative numeric solution scheme might be to use a *finite* geopotential altitude change Δh , and start from a known temperature and pressure e.g. 15°C, 1013.25mbar.

$$h \rightarrow h + \Delta h$$

$$T \rightarrow T - L\Delta h$$

$$E_s = 6.1121e^{\left\{ \left(18.678 - \frac{T}{234.5} \right) \left(\frac{T}{T+257.14} \right) \right\}}$$

$$\Delta P = -\frac{M_d g}{RT} \left(P - U \left(1 - \frac{M_v}{M_d} \right) E_s(T) \right) \Delta h$$

$$M_d = 0.02896 \text{kgmol}^{-1}$$

$$M_v = 0.01802 \text{kgmol}^{-1}$$

$$P \rightarrow P + \Delta P$$

Now the partial pressure of water vapour is typically very small compared to dry air. Therefore we *can* ignore it in pressure calculations. However, the **lapse rate** L is *significantly* affected by the presence of water vapour. A model for L is:

$$L = -\frac{dT}{dh} \quad \text{Lapse rate}$$

$$L = g \frac{1 + \frac{r\Delta H_s}{R_s T}}{c_{pd} + \frac{(\Delta H_s)^2 \frac{M_v}{M_d} r}{R_s T^2}}$$

← T is in Kelvin

Specific latent heat of vaporization of water

$$\Delta H_s = 2,501,000 \text{ J kg}^{-1}$$

Specific heat of dry air at constant pressure is:

$$c_{pd} = 1003.5 \text{ J kg}^{-1} \text{ K}^{-1}$$

Specific gas constant for dry air

$$R_s = 287 \text{ J kg}^{-1} \text{ K}^{-1}$$

$$R_s = \frac{R}{M_d} = \frac{8.314}{0.02896} \approx 287$$

$$M_d = 0.02896 \text{ kg mol}^{-1}$$

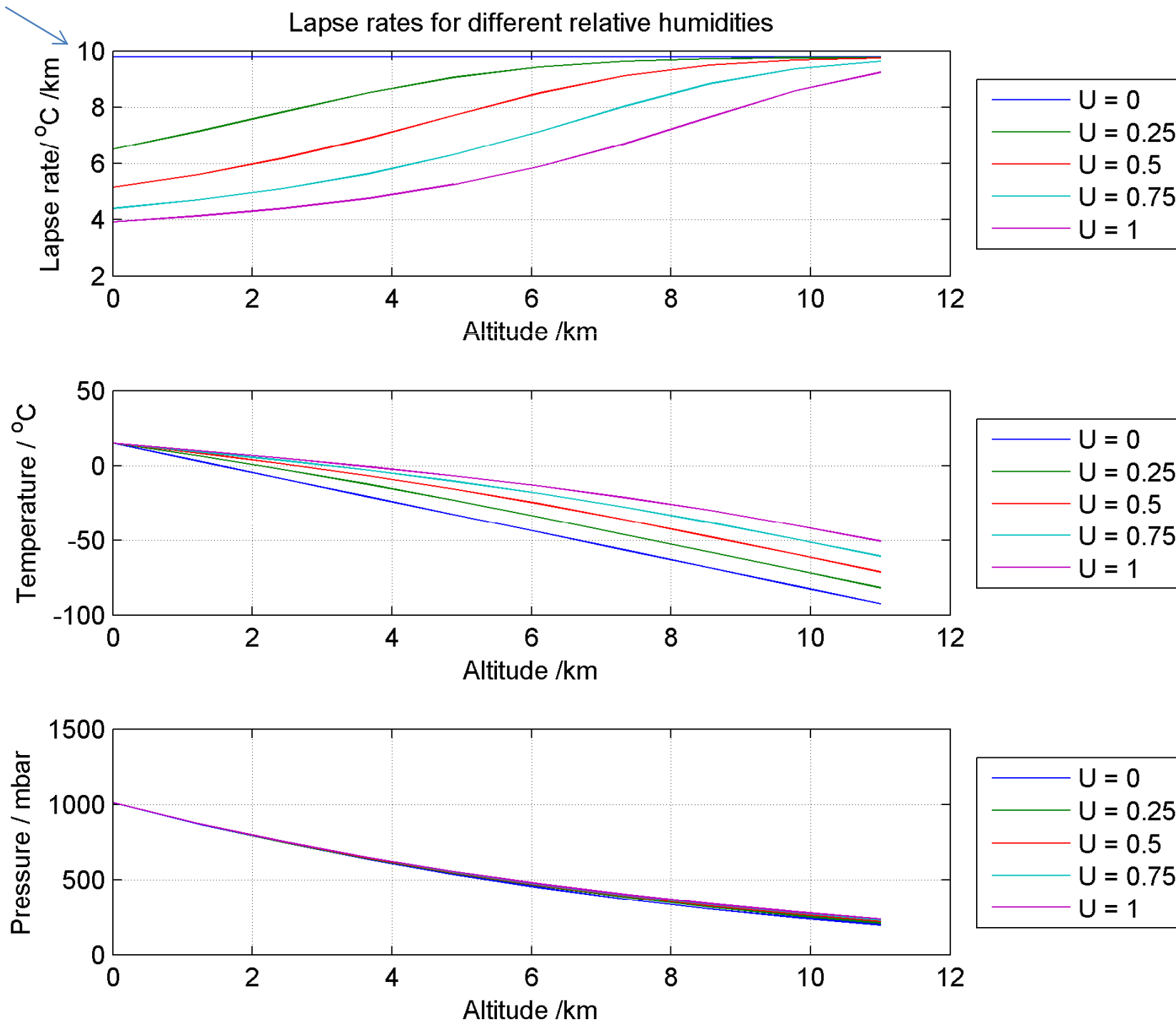
$$M_v = 0.01802 \text{ kg mol}^{-1}$$

$$g = 9.8076 \text{ ms}^{-2}$$

$$r = \frac{E_s}{P - E_s}$$

Essentially the molecular fraction of water to dry air

Hence 'rule of thumb': *dry air lapse rate is about 10° /km, wet air lapse rate is about 6° /km*



So humidity only makes a significant difference at low pressures i.e. at high altitudes. At this point the assumption of a significant relative humidity is likely to be invalid anyway!

The **dew point** is the temperature to which a given parcel of air must be cooled, at constant barometric pressure, for water vapour to condense into water.

The *August-Roche Magnus approximation* defines the dew point to be defined by the following expression in terms of relative humidity U (with values 0...1) and *ambient* air temperature T . i.e. the air parcel being cooled will be colder than the ambient air!

Note in the formula below, T is defined in degrees Celsius.

$$T_d = \frac{b \left(\ln U + \frac{aT}{b+T} \right)}{a - \ln U - \frac{aT}{b+T}}$$

$$a = 17.625$$

$$b = 243.04$$

“The dew point is the temperature at which the water vapour in a sample of air at constant barometric pressure condenses into liquid water at the same rate at which it evaporates”

http://en.wikipedia.org/wiki/Dew_point

Why clouds form

Clouds contain saturated air, i.e. where liquid water has condensed out of its vapour phase. So how is this buoyant? Surely the density must exceed the air around it?

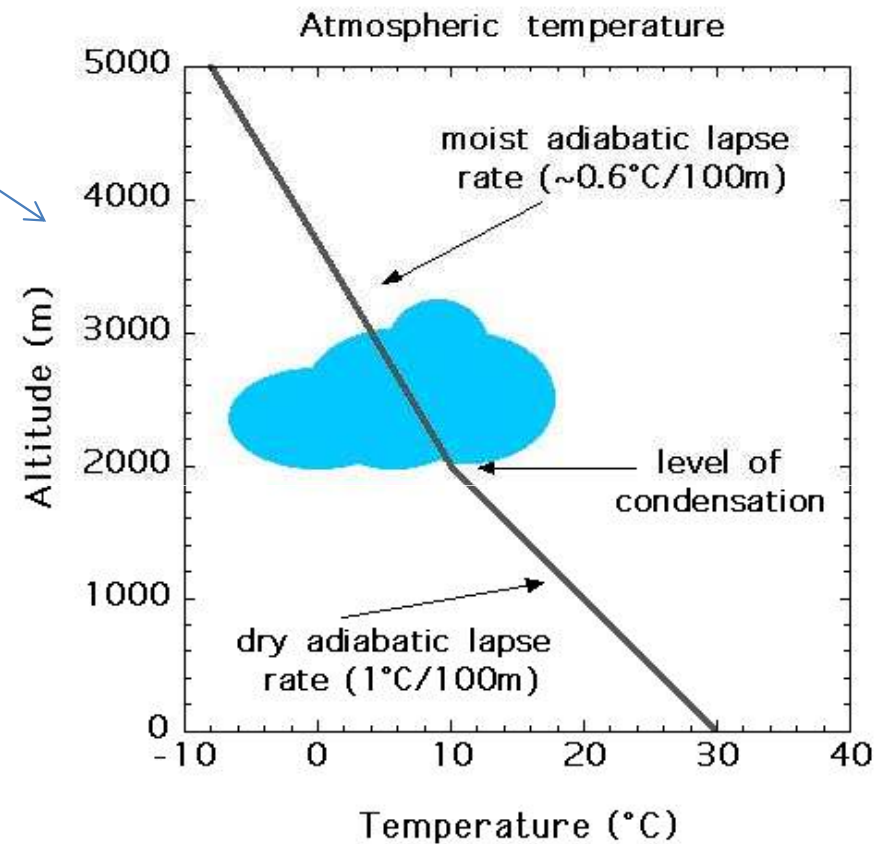
The explanation is that the air within a cloud is, on average, *warmer* than the air outside it.

This is because energy is liberated during the phase transition from vapour to liquid. A cloud therefore *contains dry air* that is *less dense* than the surrounding air. This warmer air acts as a sort of 'floatation device' for the condensed water within the cloud.

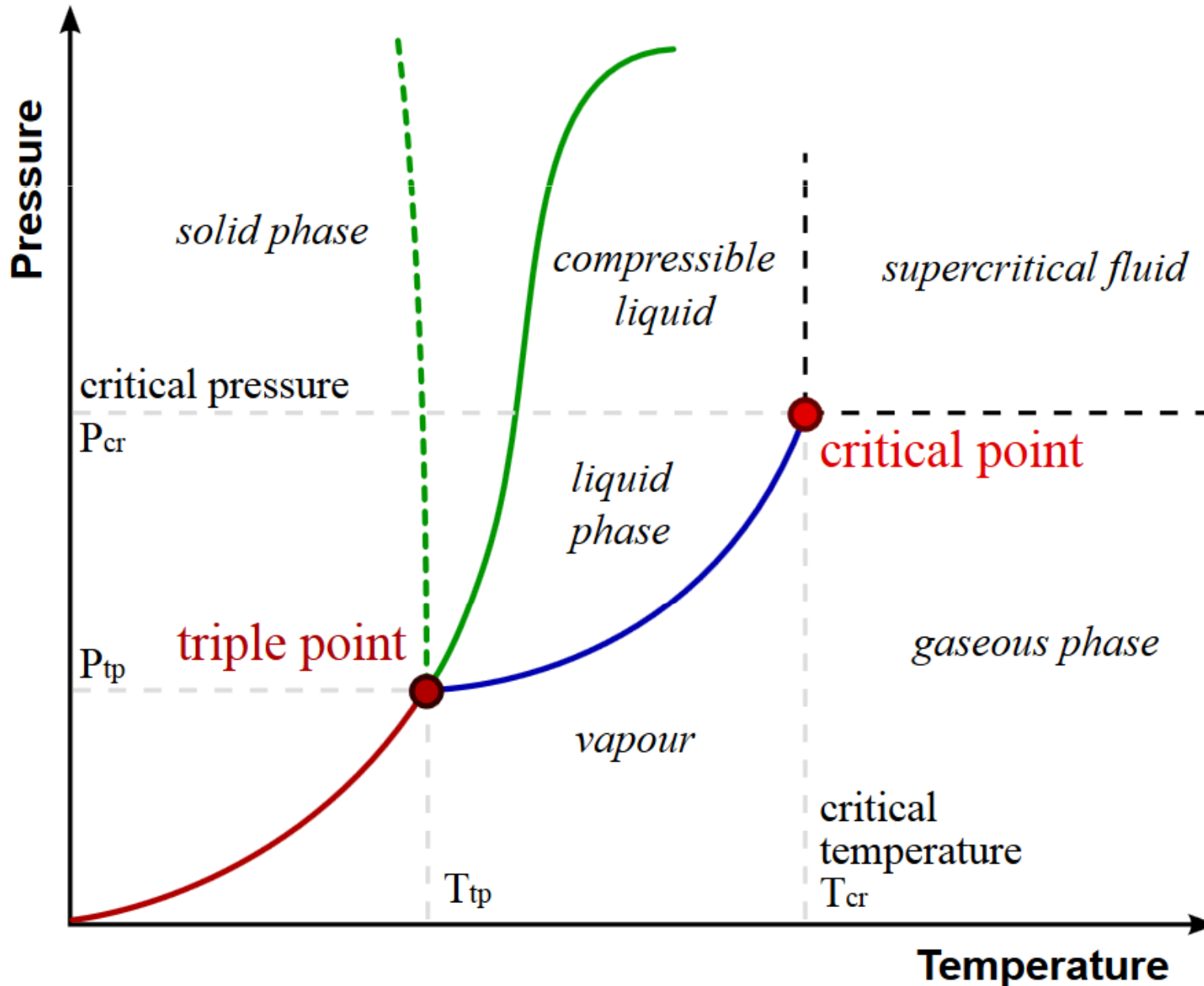
The cloud top is when the overall density inside the cloud equals to the (dry air) density outside. The cloud base is when the local air temperature reaches the dew point and condensation occurs.

One can model temperatures this by considering a *moist lapse rate for the cloud* and a *dry lapse rate for the surrounding air*.

adiabatic means
"no heat
transfer"



What about boiling point? To boil water, it must undergo a *phase transition* from liquid to gas. This requires a certain amount of heat, *the latent heat of vaporization*, to break the inter-molecular bonds inherent in the water.



The gradient of a phase transition line in the P,T diagram is given by the *Clausius-Clapeyron equation*:

Pressure in Pascals

Latent heat /Jmol⁻¹

$$\frac{dP}{dT} = \frac{\Delta H}{T\Delta V}$$

Volume change of 1 mole of substance during the phase transition

Temperature in Kelvin

In a liquid to gas transition we can assume the volume change is sufficiently large as to ignore the original fluid volume. If one assumes the resulting gas is ideal:

$$\Delta V = \frac{RT}{P}$$

Hence:

$$\frac{dP}{dT} = \frac{\Delta H}{RT^2} P$$

We can use this relationship to determine the liquid-to-gas line in the P,T diagram if the latent heat of vaporization is assumed to be temperature independent. (In reality this is not the case, but is less of an issue at low temperatures – see next slide).

$$\frac{dP}{dT} = \frac{\Delta H}{RT^2} P$$

$$\int_{P_*}^P \frac{1}{P} dP = \frac{\Delta H}{R} \int_{T_*}^T \frac{1}{T^2} dT$$

$$\ln\left(\frac{P}{P_*}\right) = -\frac{\Delta H}{R} \left(\frac{1}{T} - \frac{1}{T_*}\right)$$

$$P = P_* e^{-\frac{\Delta H}{R} \left(\frac{1}{T} - \frac{1}{T_*}\right)}$$

$$T = \left(\frac{1}{T_*} - \frac{R}{\Delta H} \ln\left(\frac{P}{P_*}\right) \right)^{-1}$$

Since the temperature corresponds to the liquid-to-gas transition line, we can therefore derive an expression for the **boiling point** of a liquid relative to ambient temperature T and pressure P , as long as one fixed boiling point T_* and corresponding pressure P_* is known.

$$T_{boil} = \left(\frac{1}{T_*} - \frac{R}{\Delta H} \ln \left(\frac{P}{P_*} \right) \right)^{-1}$$

Boiling point of water at different atmospheric pressures /mbar.

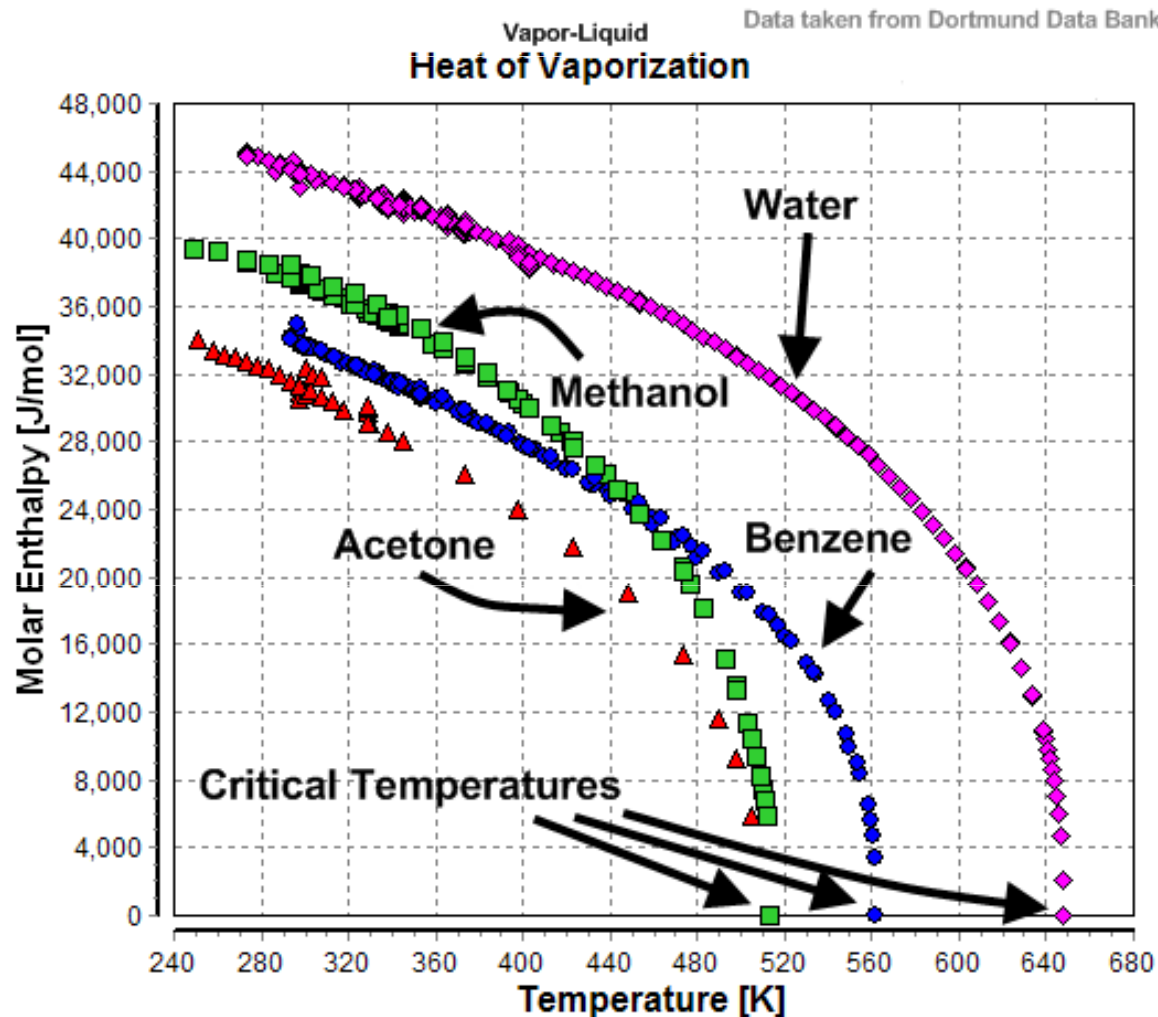
Latent heat of vaporization of water is $\Delta H = 40.7 \text{ kJ mol}^{-1}$ at 100°C and 1013.25 mbar ambient air pressure.

At 'typical' troposphere temperatures, a higher value of $45.07 \text{ kJ mol}^{-1}$ is used in some calculations, such as the lapse rate model quoted earlier.

http://en.wikipedia.org/wiki/Lapse_rate

http://en.wikipedia.org/wiki/Enthalpy_of_vaporization

http://en.citizendium.org/wiki/Heat_of_vaporization



In **summary**, we can model the variation of pressure, temperature, lapse rate boiling point and dew point with altitude using the following iterative scheme:

$$T_0 = 15^\circ\text{C} \quad T_* = 100^\circ\text{C} \quad P_0 = P_* = 1013.25\text{mbar} \quad \Delta h = 0.01\text{km}$$

$$h \rightarrow h + \Delta h \quad T \rightarrow T - L\Delta h$$

$$E_s = 6.1121e^{\left\{ \left(18.678 - \frac{T}{234.5} \right) \left(\frac{T}{T+257.14} \right) \right\}}$$

$$r = \frac{E_s}{P - E_s}$$

$$L = g \frac{1 + \frac{r\Delta H_s}{R_s T}}{c_{pd} + \frac{(\Delta H_s)^2 \frac{M_v}{M_d} r}{R_s T^2}}$$

$$\Delta P = -\frac{M_d g}{RT} \left(P - U \left(1 - \frac{M_v}{M_d} \right) E_s(T) \right) \Delta h$$

$$P \rightarrow P + \Delta P$$

$$T_{boil} = \left(\frac{1}{T_*} - \frac{R}{\Delta H} \ln \left(\frac{P}{P_*} \right) \right)^{-1}$$

$$T_{dew} = \frac{b \left(\ln U + \frac{aT}{b+T} \right)}{a - \ln U - \frac{aT}{b+T}}$$

$$a = 17.625$$

$$b = 243.04$$

Note this works out T in Kelvin

Lapse rates for different relative humidities

