A **standing wave** is formed from the *superposition* of a ‘rightwards’ and a ‘leftwards’ wave such that the **nodes** and **antinodes** of the resulting *interference pattern* don’t move. A guitar string is a good example. The essentially ‘infinite impedance’ end fixings force a *node* at *both ends* of the string. This causes a wave to reflect, *and invert,* off of the ends. The superposition of this and the incoming wave creates the standing wave pattern.

$$\psi(x,t) = \underbrace{A\sin(kx-\omega t)}_{\text{incoming wave}} - \underbrace{A\sin(-kx-\omega t)}_{\text{reflected wave}}$$

-ve sign means the wave is inverted following reflection

$$\psi(x,t) = A\sin kx\cos(-\omega t) + A\cos kx\sin(-\omega t) - \{A\sin(-kx)\cos(-\omega t) + A\cos(-kx)\sin(-\omega t)\}$$

$$\psi(x,t) = A\sin kx\cos\omega t - A\cos kx\sin\omega t - \{-A\sin kx\cos\omega t - A\cos kx\sin\omega t\}$$

$$\psi(x,t) = A\sin kx\cos\omega t - A\cos kx\sin\omega t + A\sin kx\cos\omega t + A\cos kx\sin\omega t$$

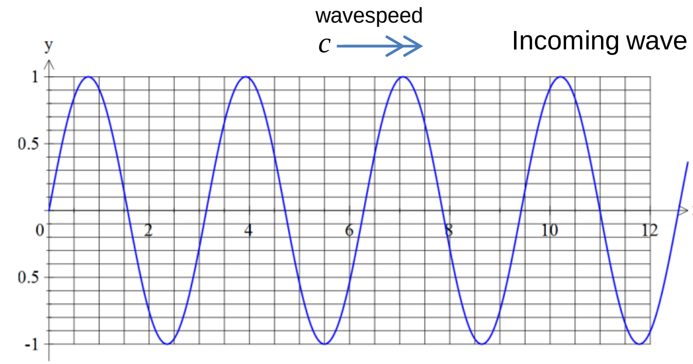
$\sin(A+B) = \sin A\cos B + \cos A\sin B$

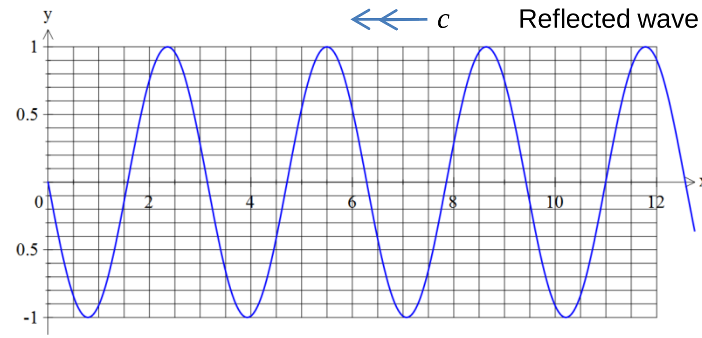
$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$$\boxed{\psi(x,t) = 2A\sin kx\cos\omega t}$$

The standing wave pattern separates the *spatial* from the *temporal* parts.




For a standing wave on a string with **nodes** at either end, a whole number ($n$) of half wavelengths must add up to the string length:

$$n\tfrac{1}{2}\lambda = L \quad \therefore \lambda = \frac{2L}{n} \quad \therefore k = \frac{2\pi}{\lambda} = \frac{n\pi}{L}$$

$$\boxed{\therefore \psi(x,t) = \psi_0 \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{2\pi t}{T}\right)}$$

$\psi_0$ is the maximum wave amplitude

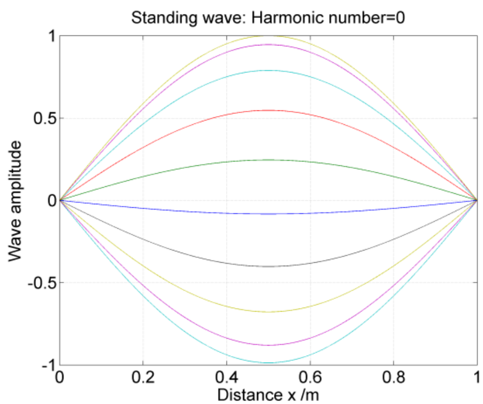
Wave time period $T$

For the graphs below, the plot is of $\psi_0 \sin\left(\frac{n\pi}{L}x\right)$ vs $x$

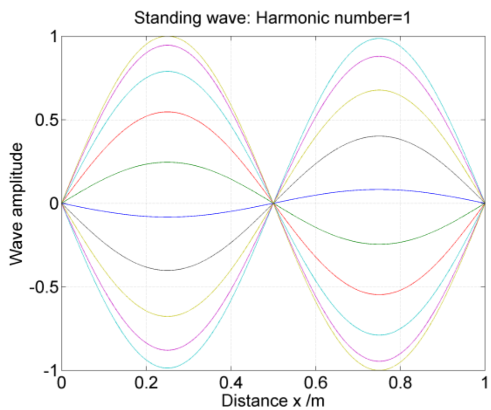
with amplitude scaled by the time dependent factor $\cos\left(\frac{2\pi}{T}t\right)$

$$c = \frac{\lambda}{T} \quad \therefore \cos\left(\frac{2\pi t}{T}\right) = \cos\left(\frac{2\pi ct}{\lambda}\right) = \cos\left(\frac{n\pi ct}{L}\right)$$

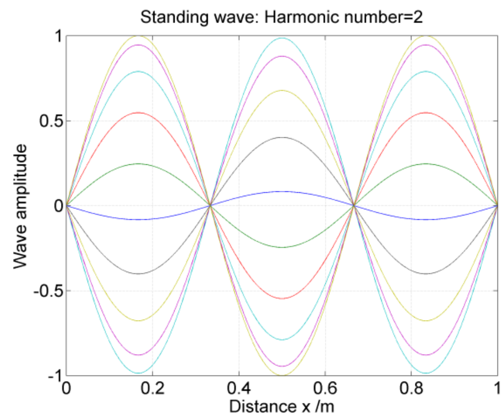
$n=1$ **Fundamental**




$n=2$ **1st harmonic**



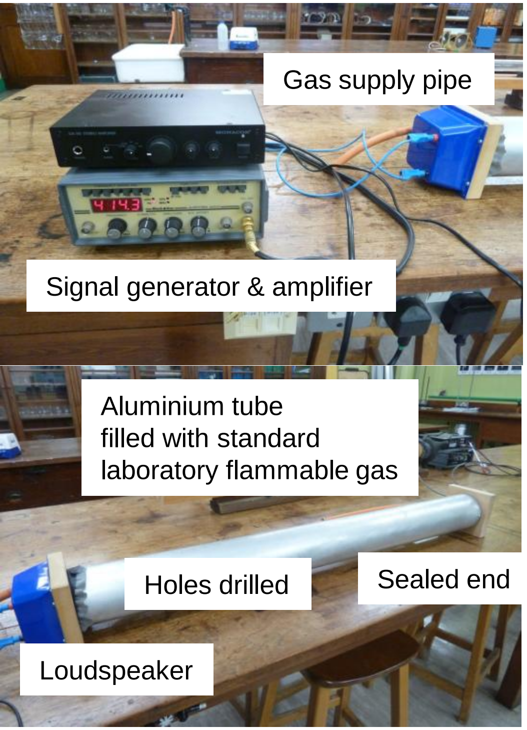

$n=3$ **2nd harmonic**




i.e. nodes and antinodes remain in the *same* spatial positions, but the wave amplitude varies in an oscillatory fashion with period $T$.

**Rubens' tube.** Closed tube filled with flammable gas, driven at one end by a loudspeaker.

Heinrich Rubens
1865-1922




*Pressure antinodes* are at the loudspeaker *and* closed ends of the tube when a standing wave is formed.

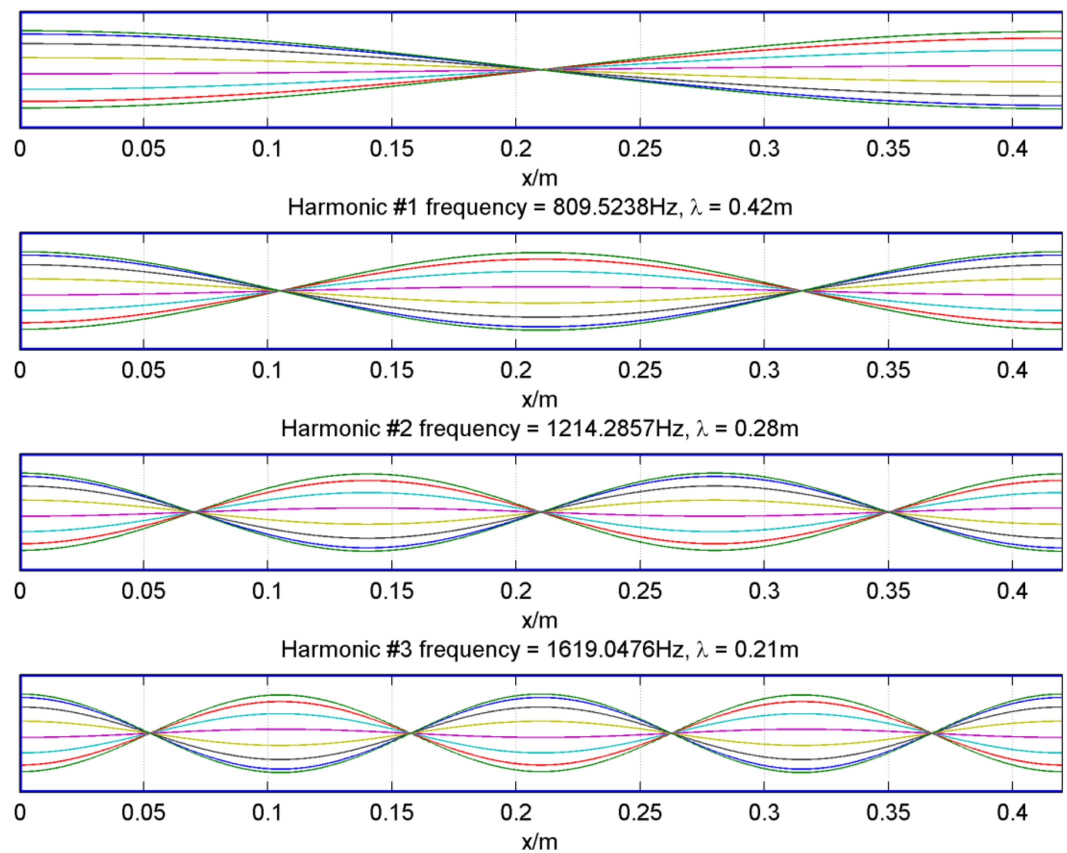
This means a *whole number of half wavelengths must equal the tube length* in order for a standing wave to be formed.

$$f_n = \frac{c}{2L}n$$

$$\lambda_n = \frac{c}{f_n} = \frac{2L}{n}$$

$$\therefore p(x,t) = p_{\max}\cos\left(2\pi\frac{x}{\lambda_n}\right)\cos\left(2\pi f_n t\right)$$

Maximum flame heights correspond to the maximum pressure difference between the gas in the tube and ambient air pressure i.e. at **pressure antinodes**




Pressure standing waves in a Rubens' tube evaluated at various times during a period.

Allow gas to enter tube and then light the holes with a match or lit splint. Move along tube to alight the fifty or so holes.

As a sound wave is set up in the tube, the pattern of pressure nodes and antinodes will result in different heights of flames.


Open end of tube. Sawdust inside it reveals the nodes and antinodes of sound waves

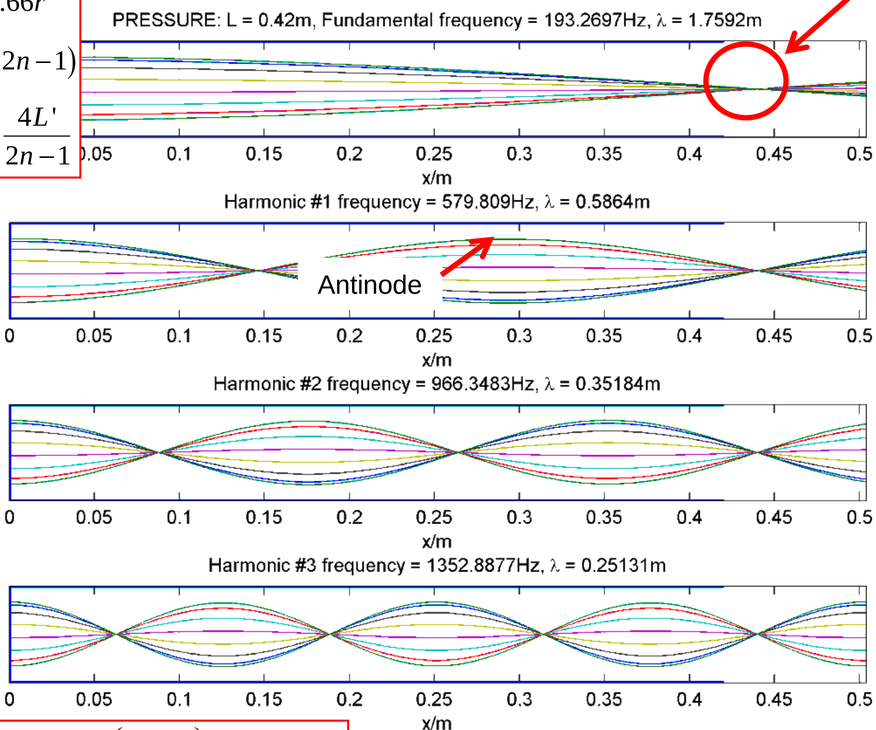
For a **tube closed at one end and open at the other**, there is a *pressure antinode* at the *closed end*, and a *pressure node at the end open to the atmosphere* (which fixes the pressure there). This means to form a standing wave, the tube length must be an *odd number of quarter-wavelengths.*

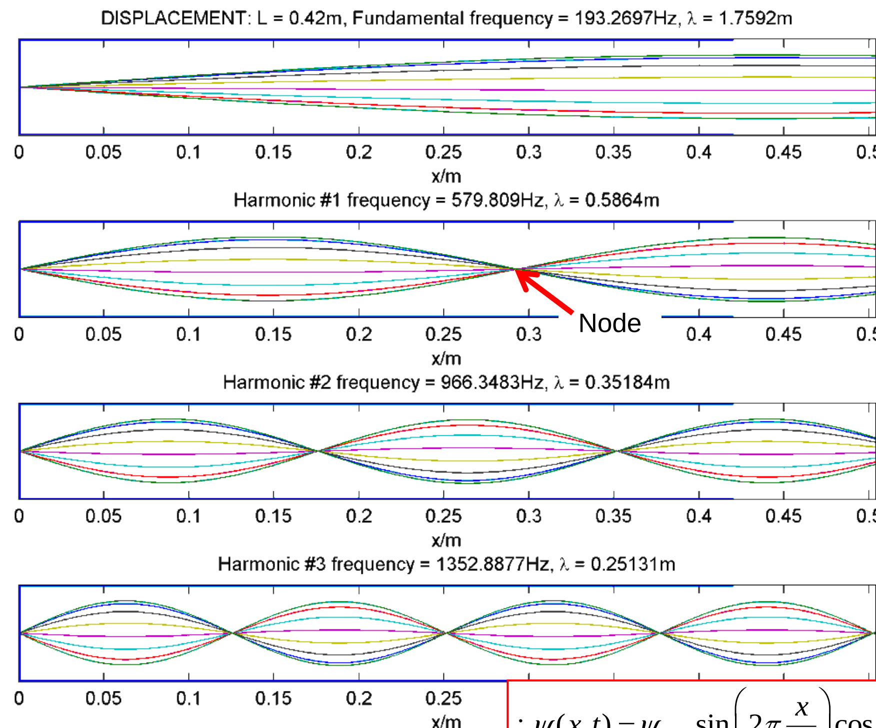
**End correction:** effect of tube vibrating air causes effective length to be increased by about 0.66*r* where *r* is the tube radius

$$L' = L + 0.66r$$

$$f_n = \frac{c}{4L'}(2n-1)$$

$$\lambda_n = \frac{c}{f_n} = \frac{4L'}{2n-1}$$




Pressure standing wave

$$\therefore p(x,t) = p_{\max} \cos\left(2\pi \frac{x}{\lambda_n}\right)\cos(2\pi f_n t)$$

Note *air pressure antinodes* are *air displacement nodes* and vice versa

$$\therefore \psi(x,t) = \psi_{\max} \sin\left(2\pi \frac{x}{\lambda_n}\right)\cos(2\pi f_n t)$$

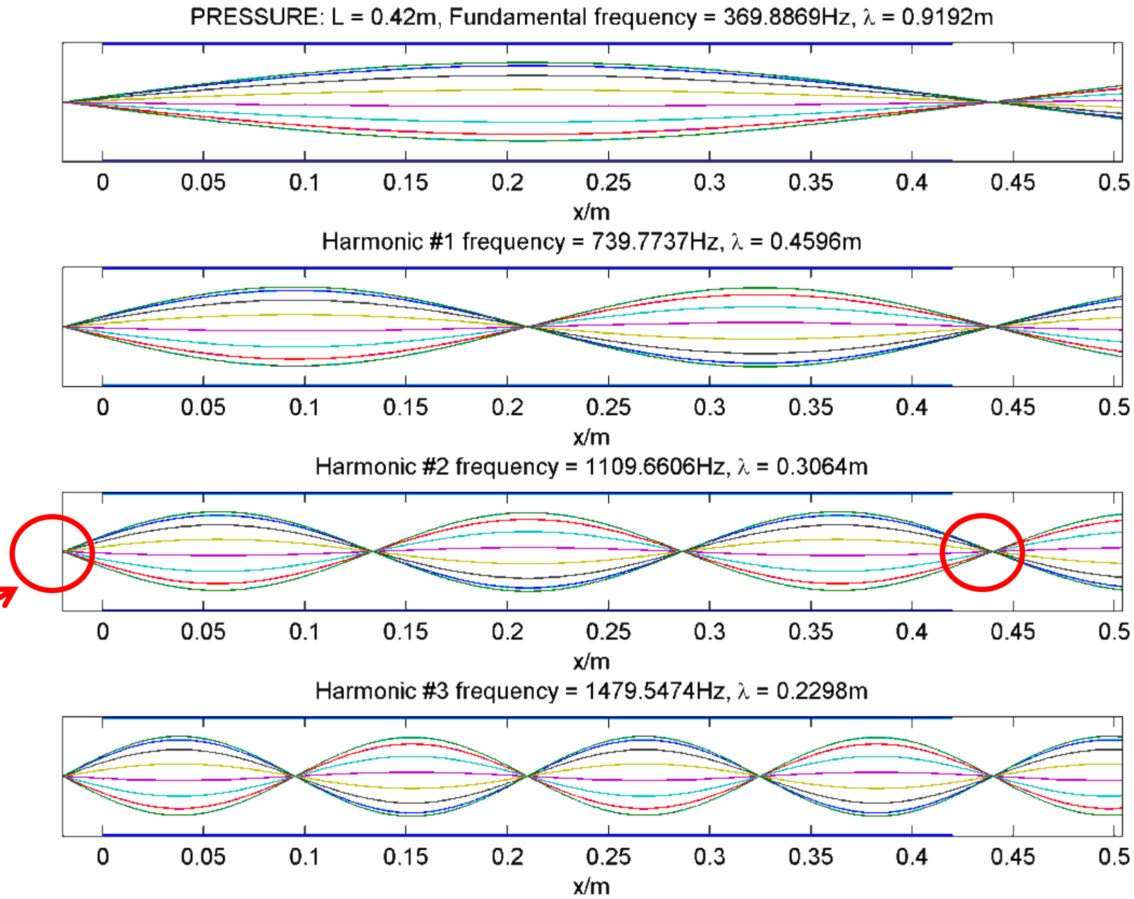
**Pan pipes** are examples of tubes *open at both ends.* To make the loudest sounds a standing wave needs to be established in the air column. Since both ends of the pipe are connected to atmospheric pressure, there must be a pressure *antinode at both ends*. This means the *tube length must be a whole number of half wavelengths* in order for a standing wave to be established.

To account for the vibration of the air by the tube, we apply an empirical **end correction** to *both* ends.

$\delta \approx 0.66r$

$L \to L + 2\delta$     $r$ is the tube radius

Pressure standing waves in a tube open at both ends, evaluated at various times during a period.




Hence to characterize the standing waves:

$$\delta \approx 0.66r$$
$$L' = L + 2\delta$$
$$f_n = \frac{c}{2L'}n$$
$$\lambda_n = \frac{c}{f_n} = \frac{2L'}{n}$$
$$p(x,t) = p_{\max}\sin\left(2\pi\frac{x+\delta}{\lambda_n}\right)\cos\left(2\pi f_n t\right)$$

End corrections