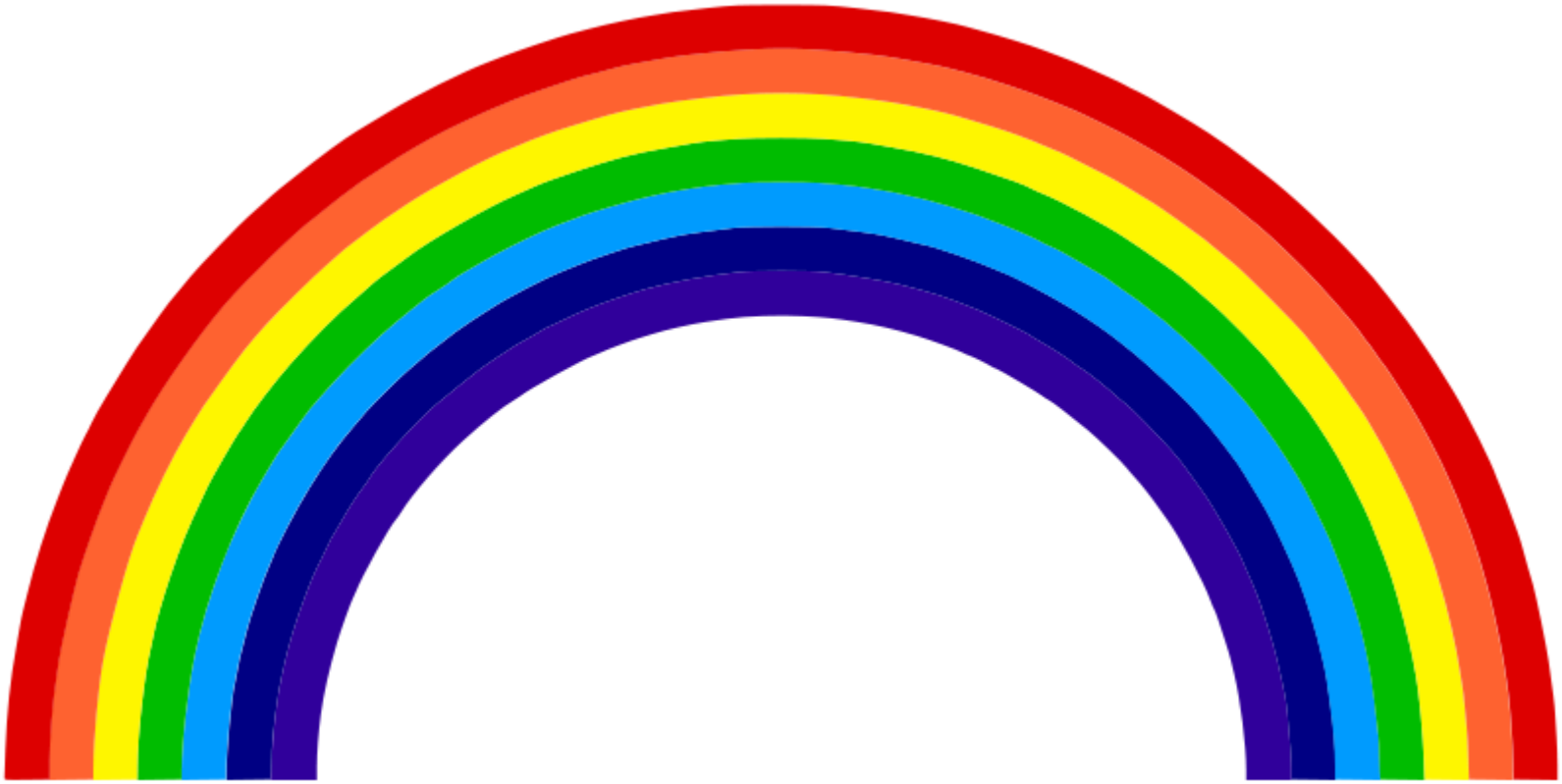


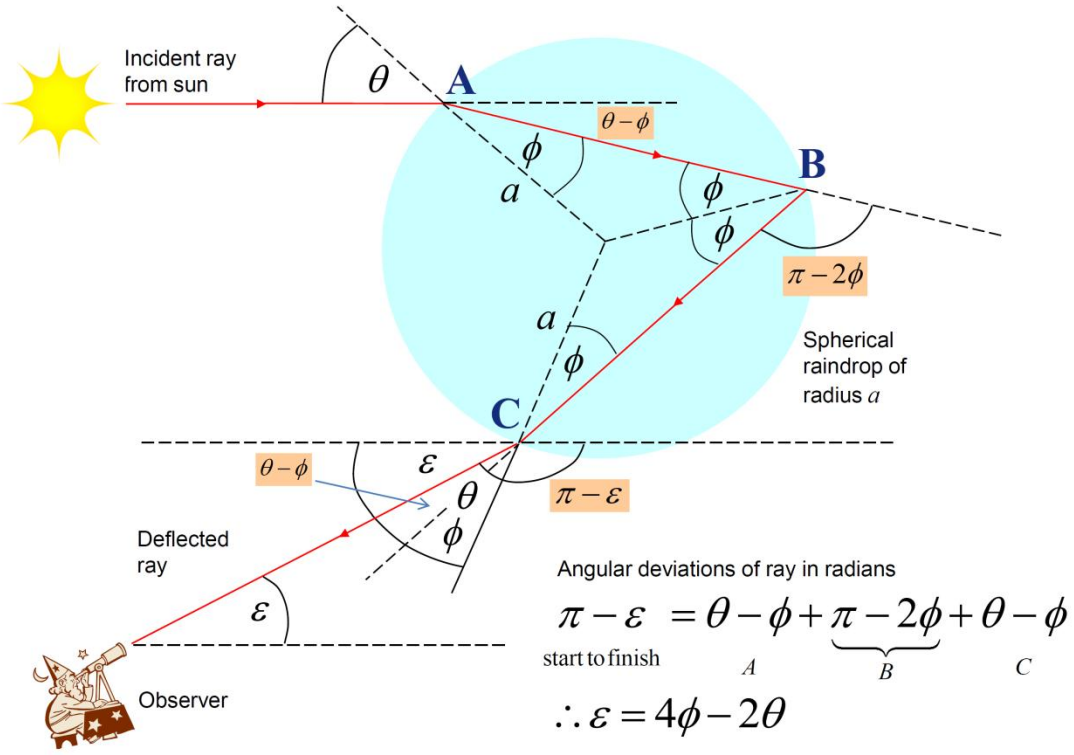
The Subtlety of Rainbows

Dr Andrew French

Summary

- When I worked in the **radar systems simulation & modelling** group of BAE Systems we often constructed **mathematical models** of physical processes (such as reflection of microwaves off aeroplanes, rain, sea…) in order to run accurate **computer simulation**s which allowed us to minimise the amount of (expensive) tests using real radar equipment.
- The example here of a mathematical model of a **rainbow** aims to help us to understand how simple principles of **physics** can be used to predict observable quantities such as the *shape*, *angular width* and *order of colours.*
- All models are built upon **assumptions** - essentially what set of scientific laws can we apply to the situation.

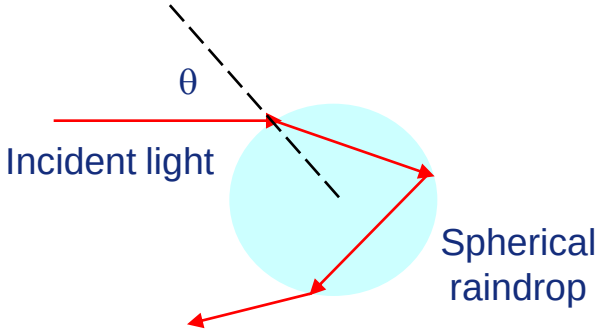
Descartes theory of the rainbow



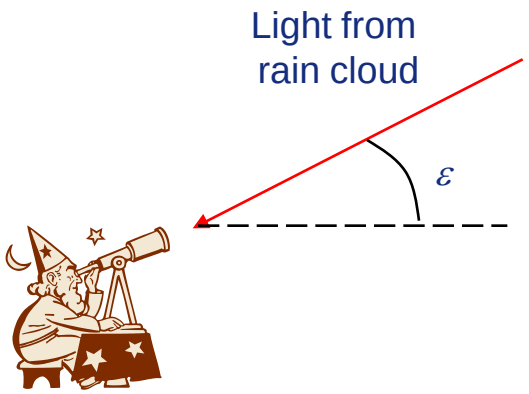
'des cartes postal'!

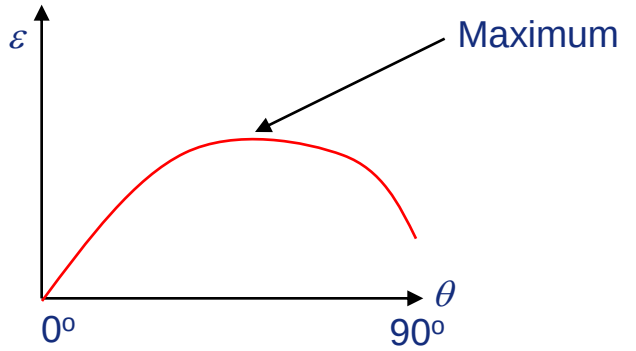

https://en.wikipedia.org/wiki/Rainbow
Primary rainbow
Secondary rainbow
Alexander's dark band
Note the colour order is swapped for primary and secondary rainbows!

Descartes theory of the rainbow - assumptions



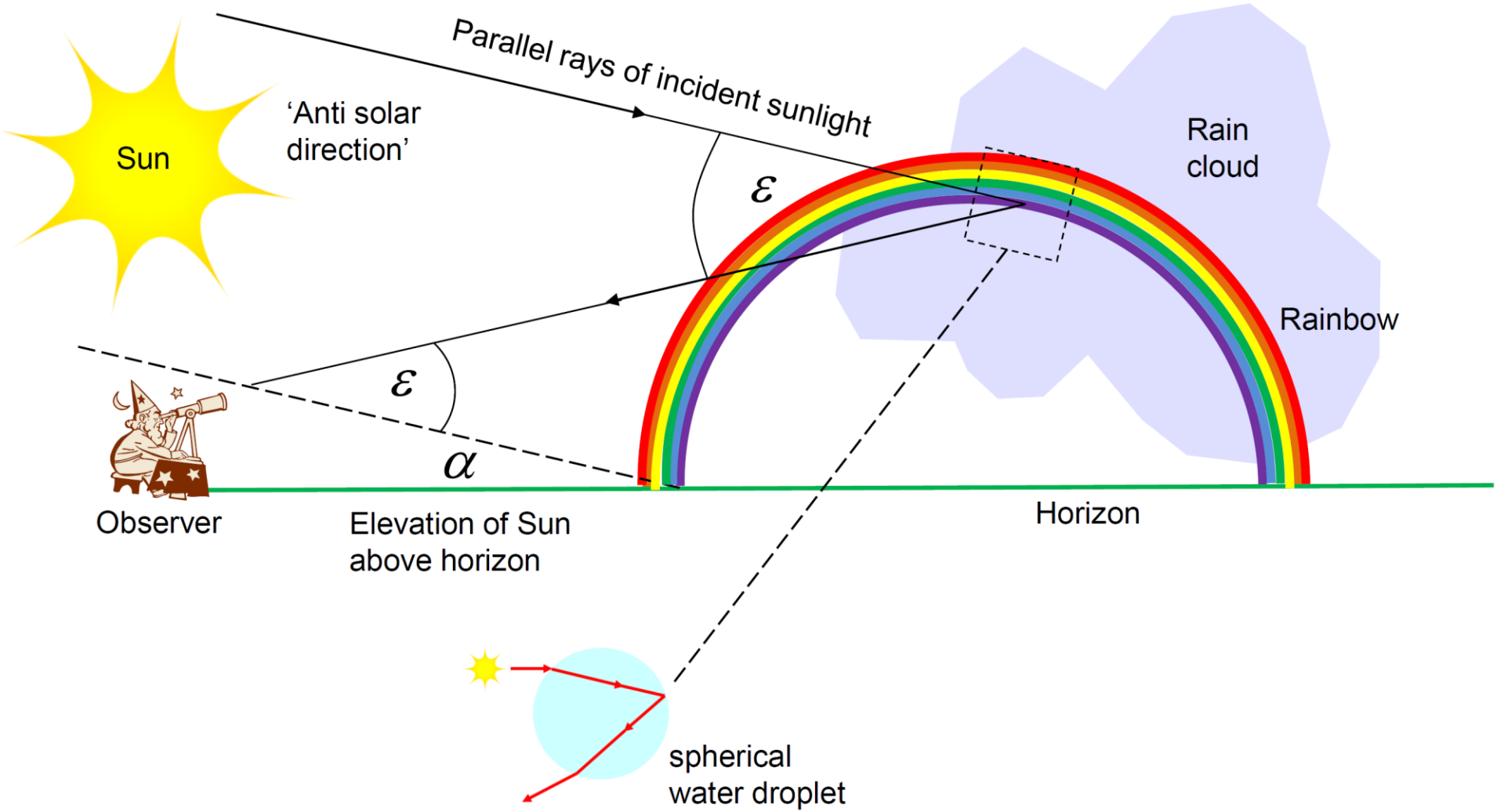
- Light is **internally reflected** off the interior of **spherical** raindrops.
- The wavelength of light is much smaller than the dimensions of the raindrop, so **interference effects** can be ignored.
- The mathematical relation between the angle that light is bent by the raindrops and the angle of incidence to the raindrop, has an 'extremum'.* This results in the **focussing** of light of particular wavelengths into particular angles.



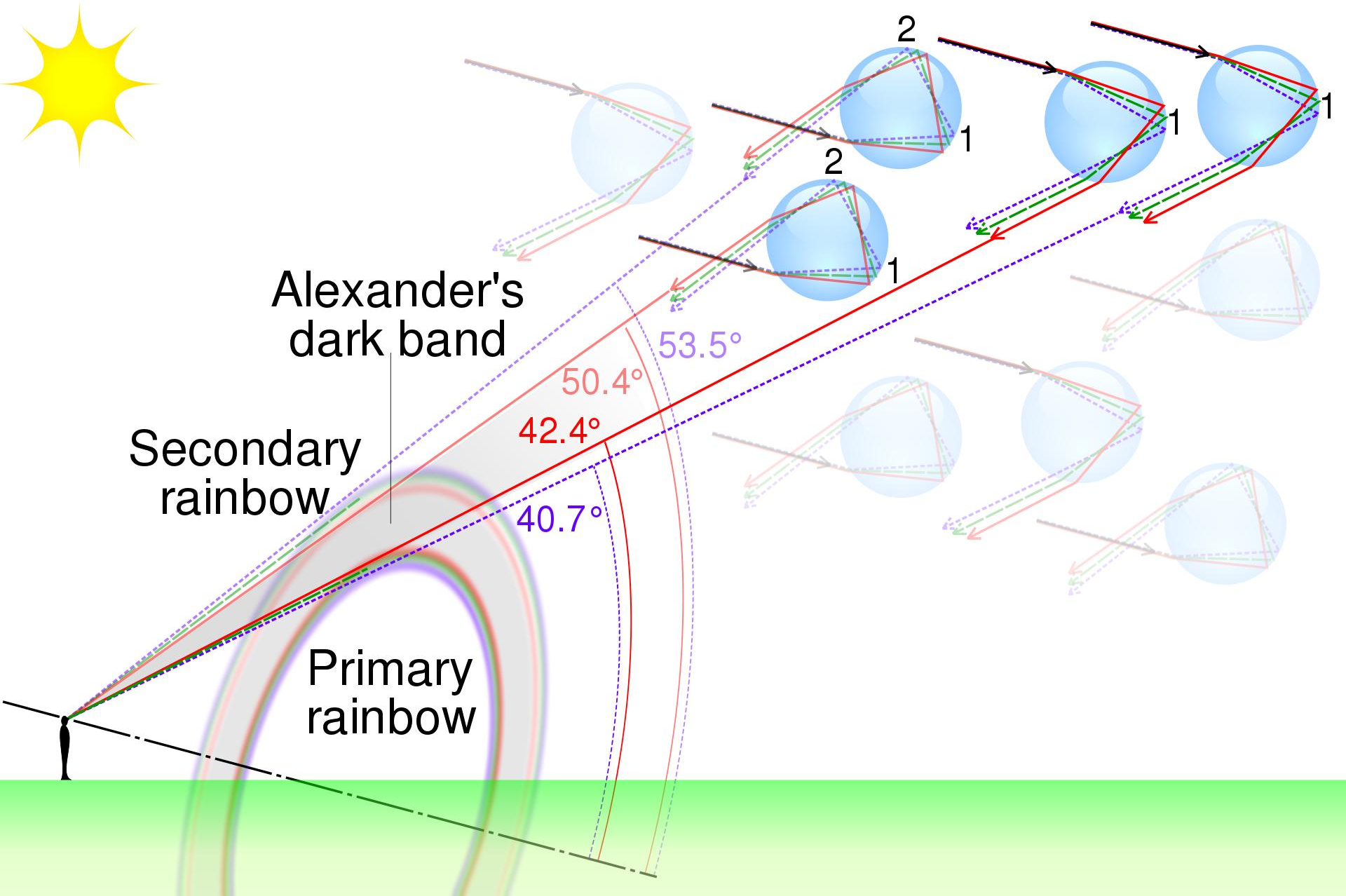


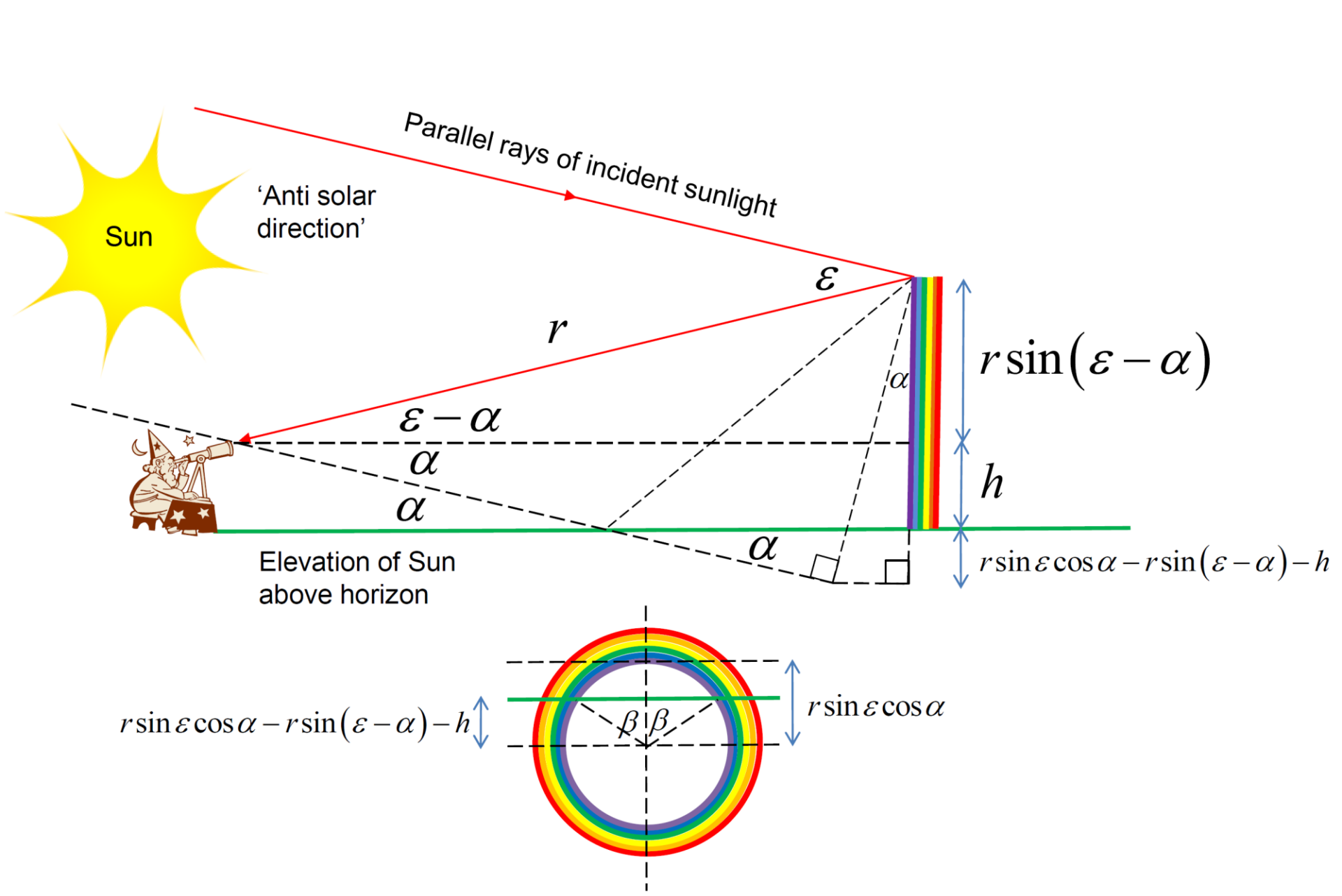
*max or min.

Schematic of a rainbow

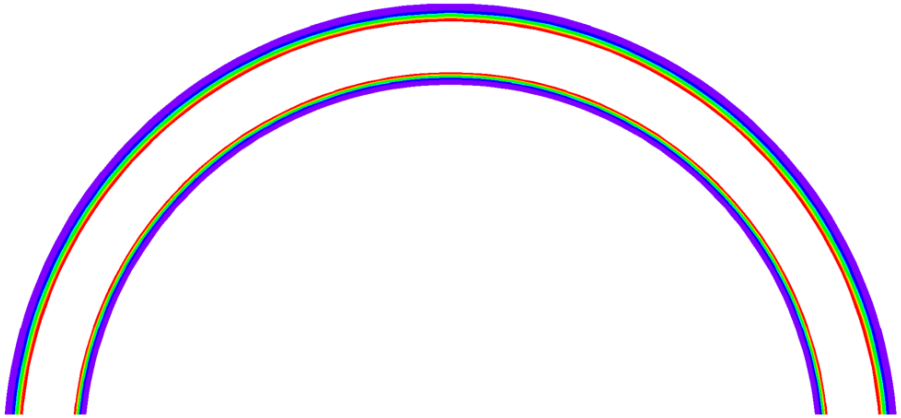


https://en.wikipedia.org/wiki/Rainbow



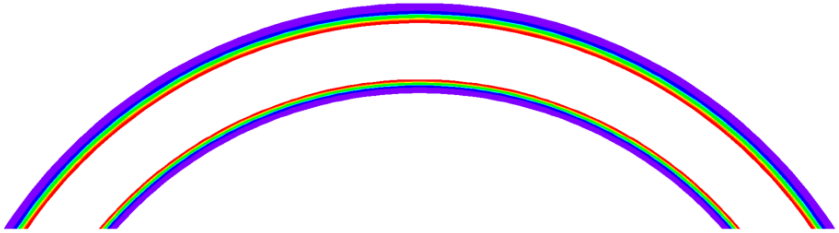

Sun
'Anti solar direction'
Parallel rays of incident sunlight
ε
r
$\varepsilon - \alpha$
α
α
α
α
$r\sin(\varepsilon-\alpha)$
h
$r\sin\varepsilon\cos\alpha - r\sin(\varepsilon-\alpha) - h$
Elevation of Sun above horizon
$r\sin\varepsilon\cos\alpha - r\sin(\varepsilon-\alpha) - h$
β
β
$r\sin\varepsilon\cos\alpha$

Solar angle α=5^{o}

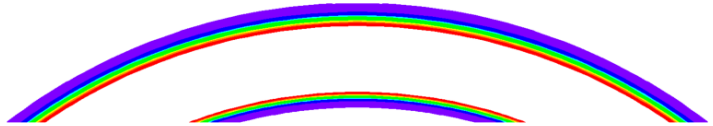


Solar angle α=20^{o}

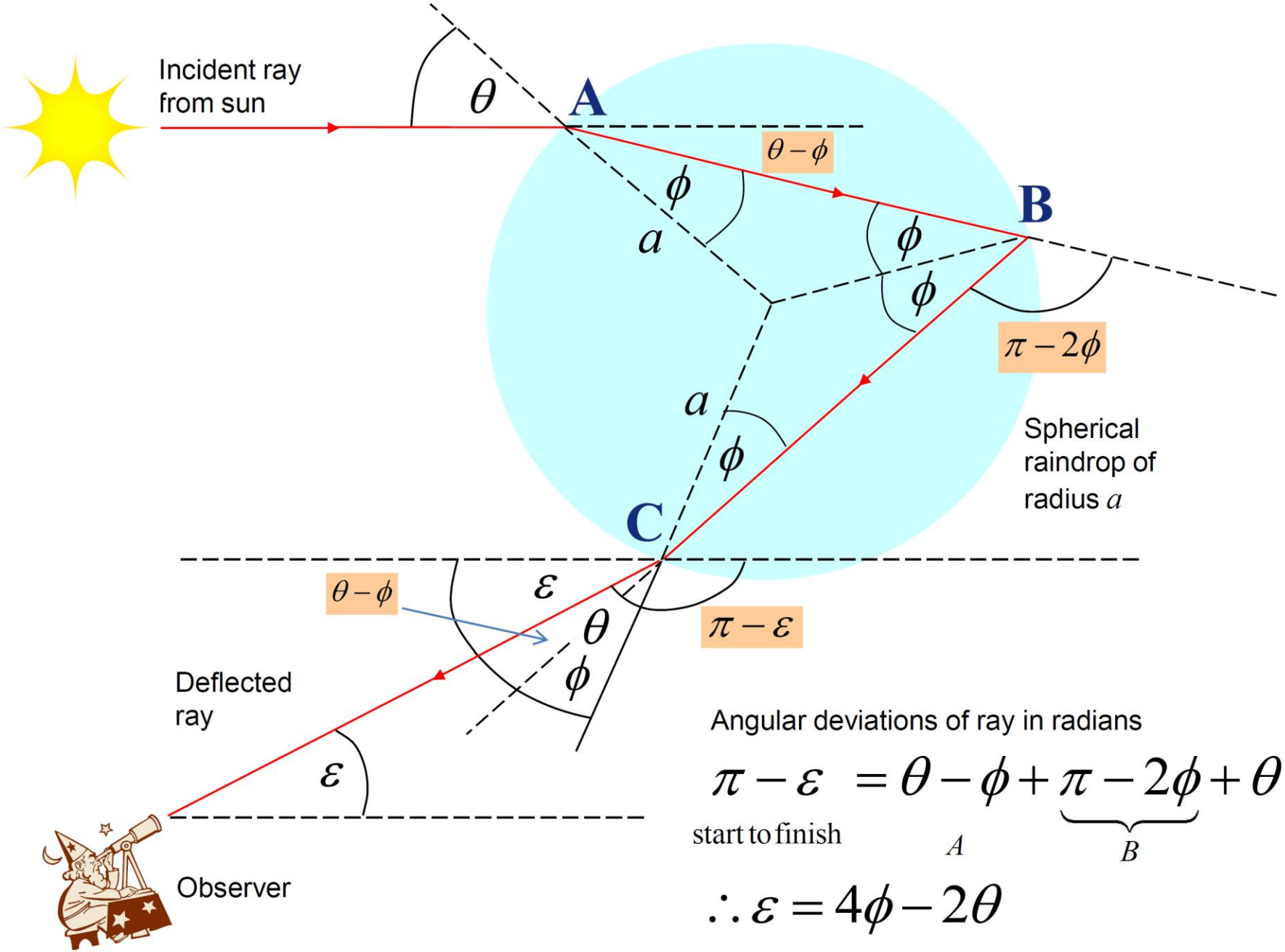
Solar angle α=30^{o}



Solar angle α=40^{o}



Primary rainbow – a single internal reflection



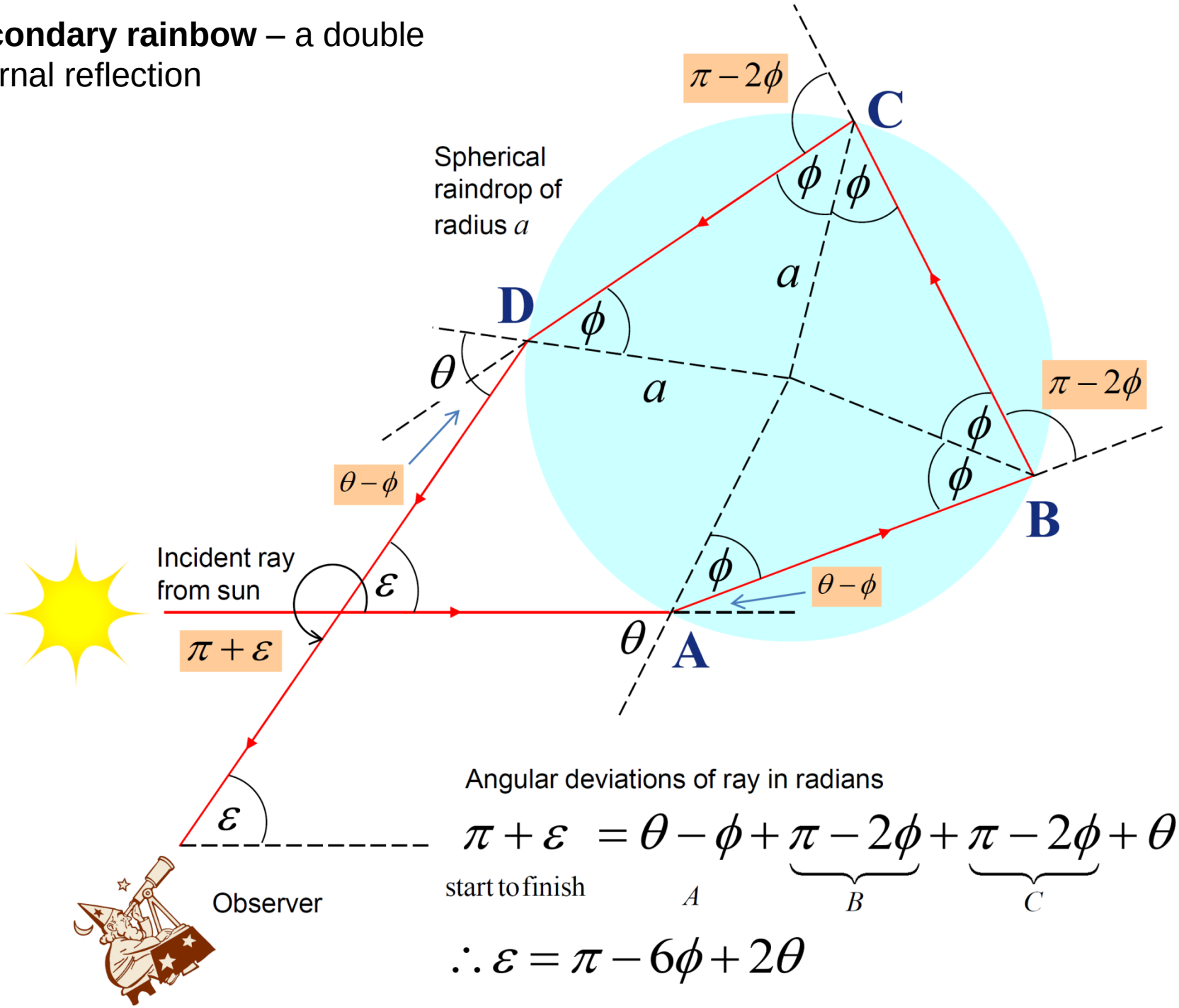
Angular deviations of ray in radians

$$\pi - \varepsilon = \theta - \phi + \underbrace{\pi - 2\phi}_{B} + \theta - \phi$$

start to finish A C

$$\therefore \varepsilon = 4\phi - 2\theta$$

Secondary rainbow – a double internal reflection



Angular deviations of ray in radians

$$\pi + \varepsilon = \theta - \phi + \underbrace{\pi - 2\phi}_{B} + \underbrace{\pi - 2\phi}_{C} + \theta - \phi$$

start to finish $\quad A \qquad\qquad D$

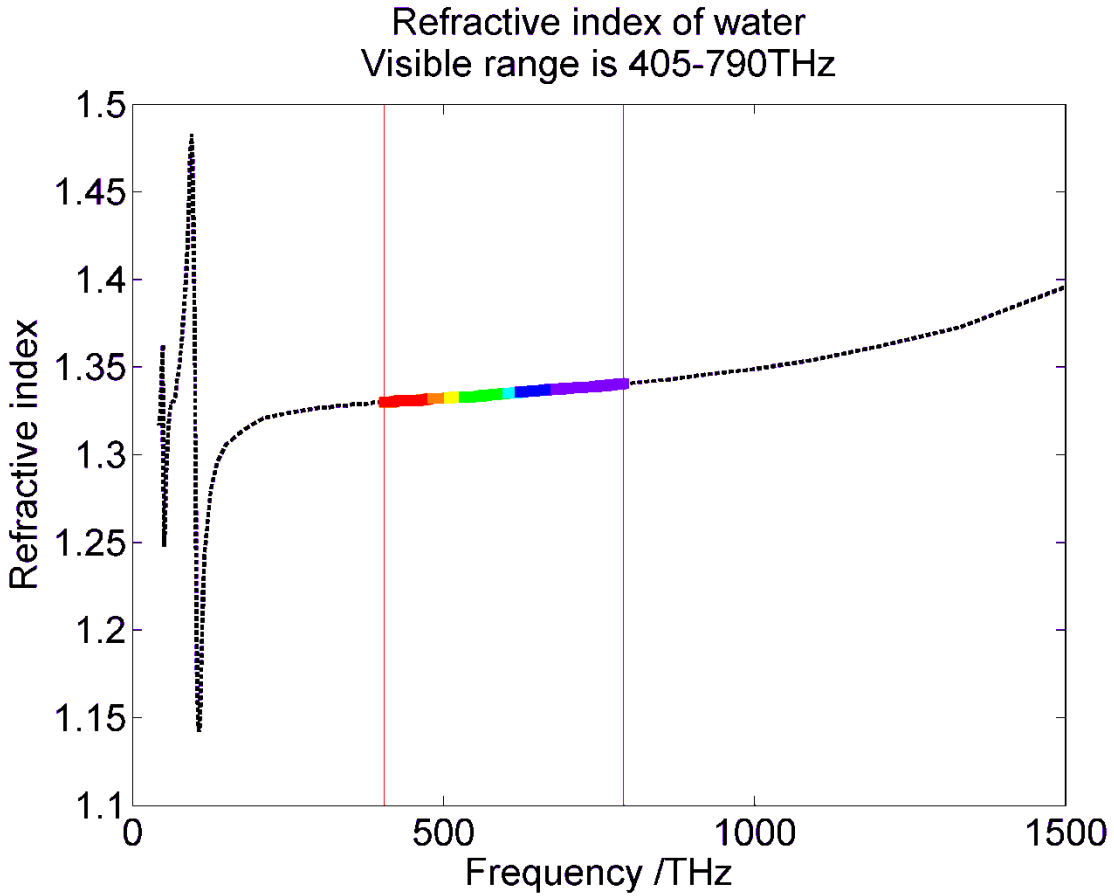
$$\therefore \varepsilon = \pi - 6\phi + 2\theta$$

Refractive index n

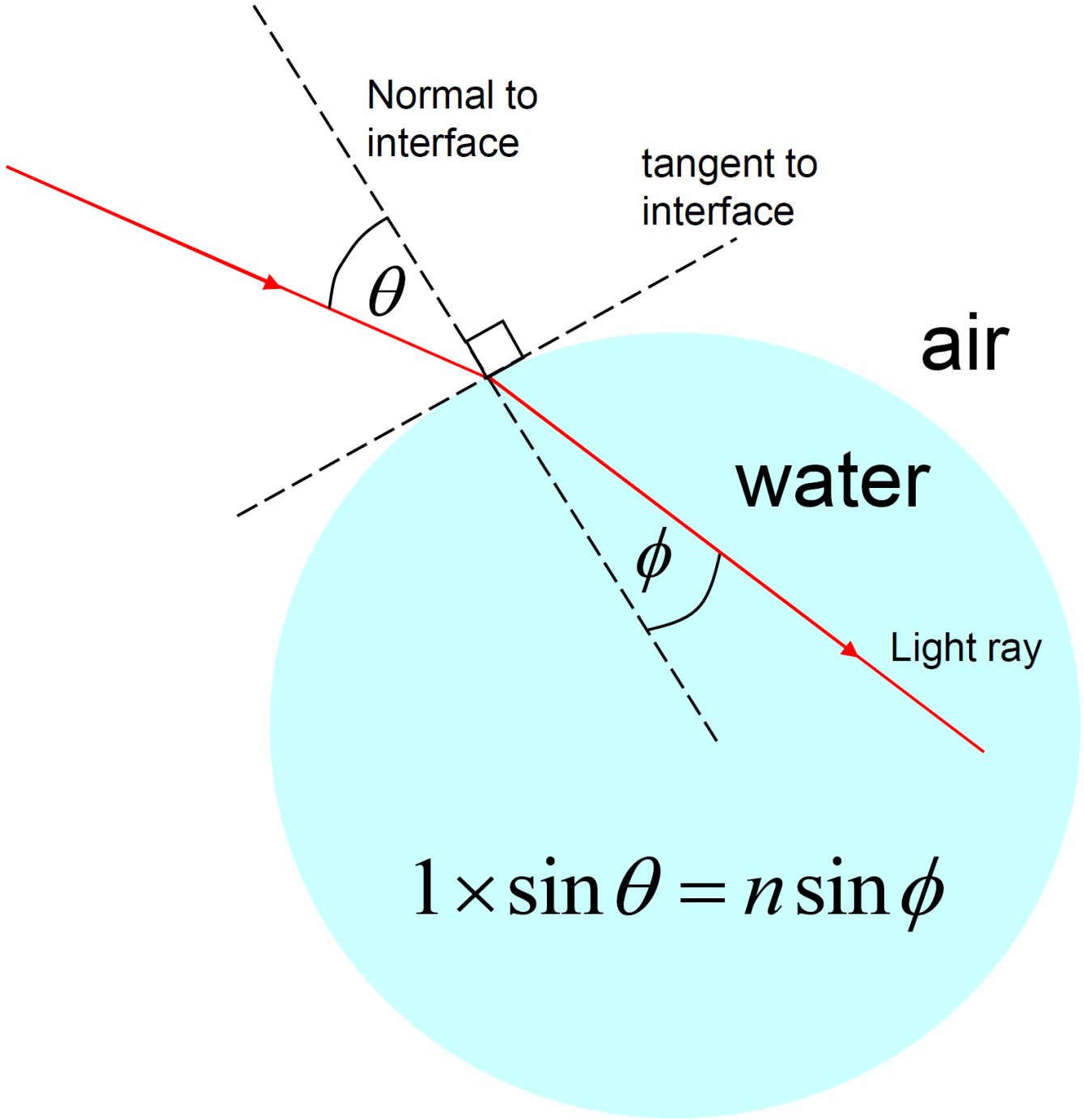
$$n = \frac{\text{speed of light in a vacuum}}{\text{speed of light in medium}}$$

$n \approx 1$ air

$n \approx 1.34$ water

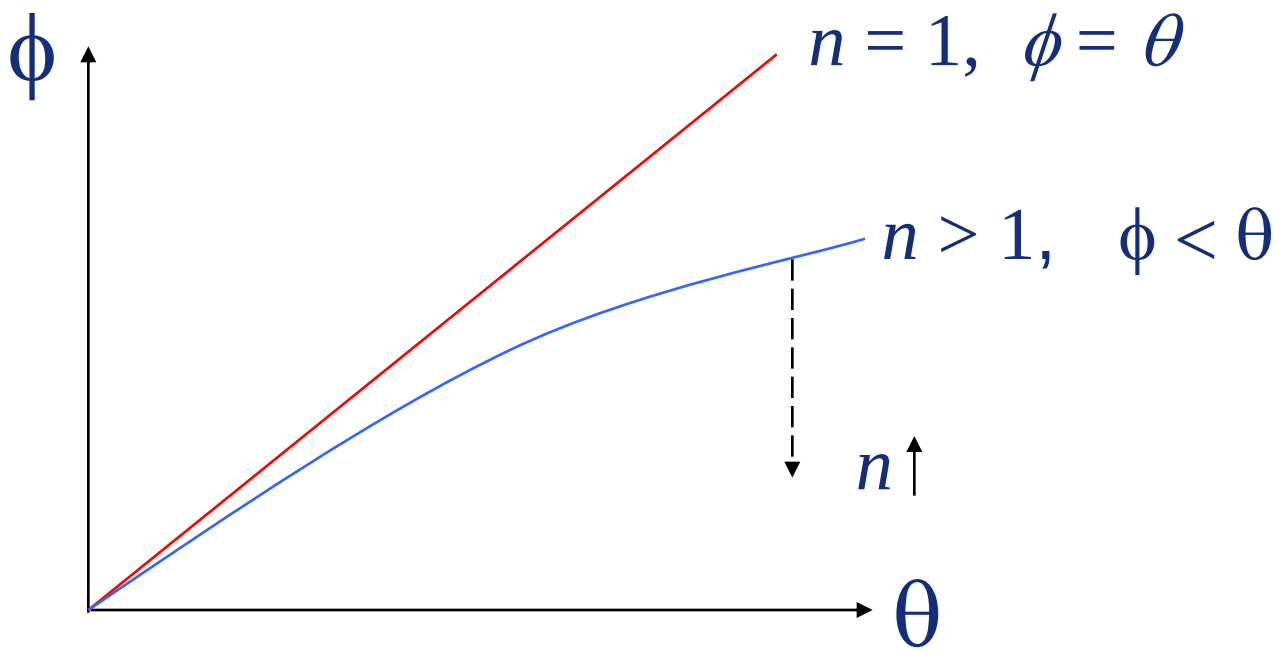


Note n often varies with the frequency of light

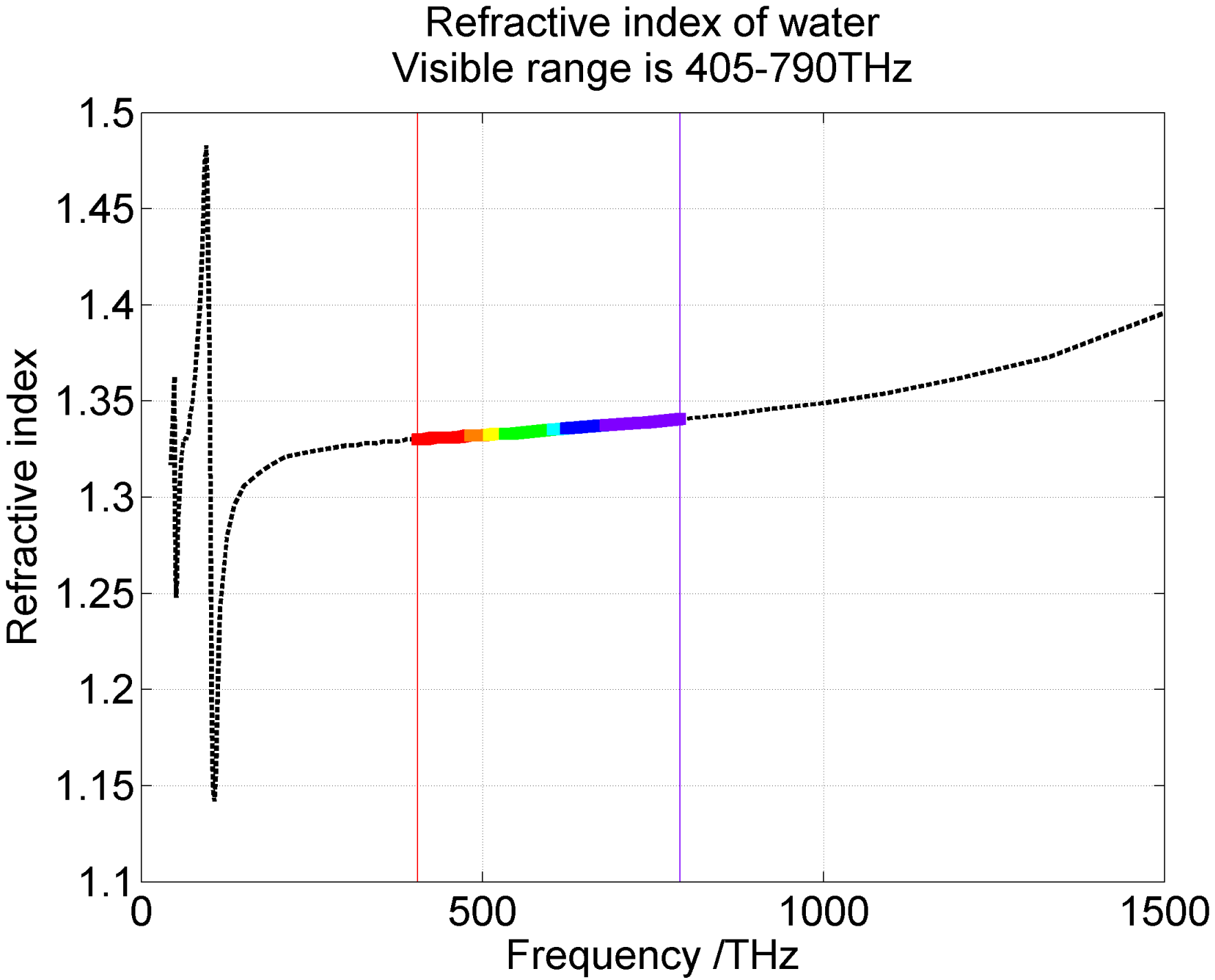

Normal to
interface
tangent to
interface
θ
air
water
ϕ
Light ray
$1 \times \sin\theta = n \sin\phi$

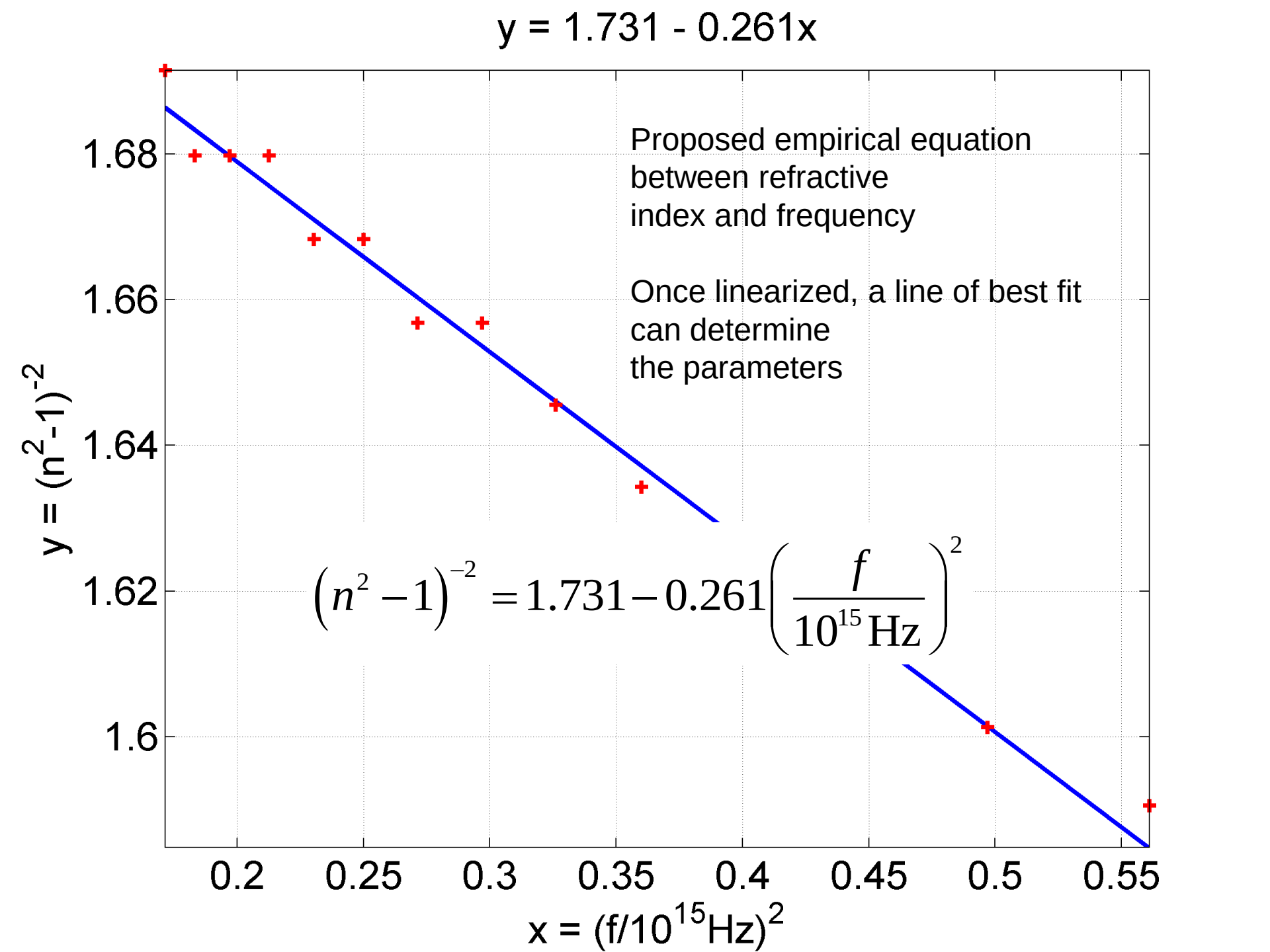
Snell's law of refraction

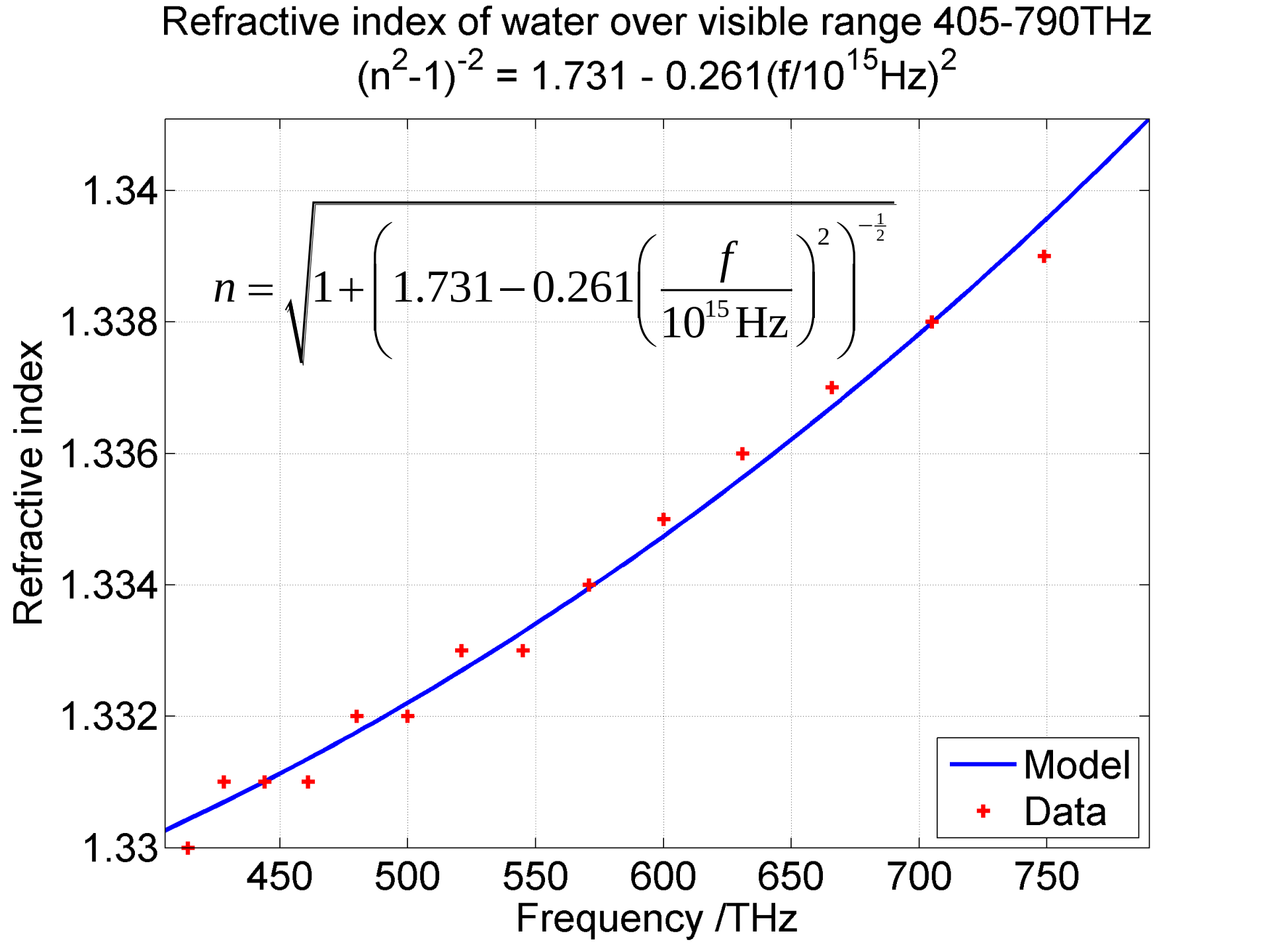
$$1 \times \sin\theta = n \sin\phi$$

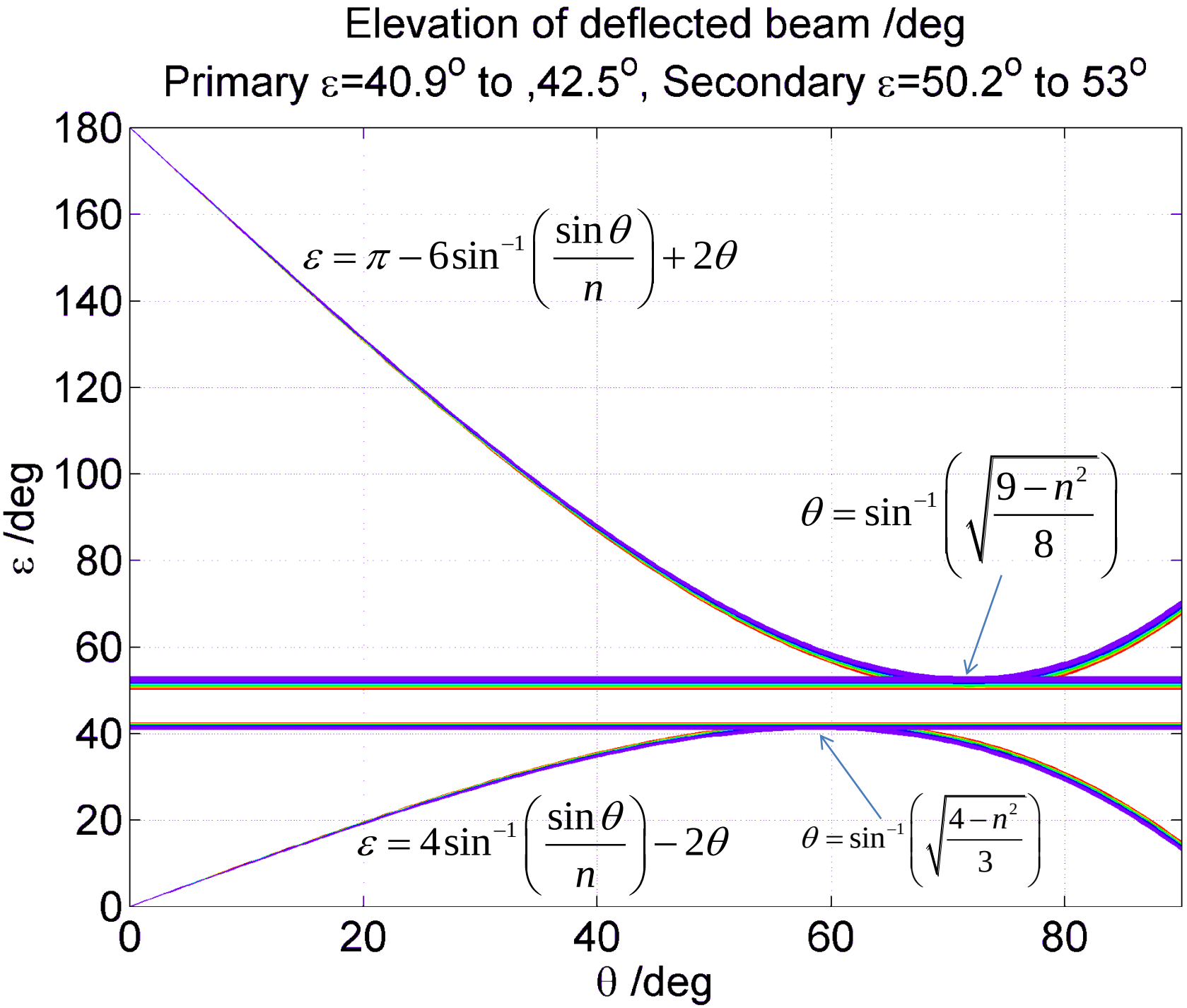


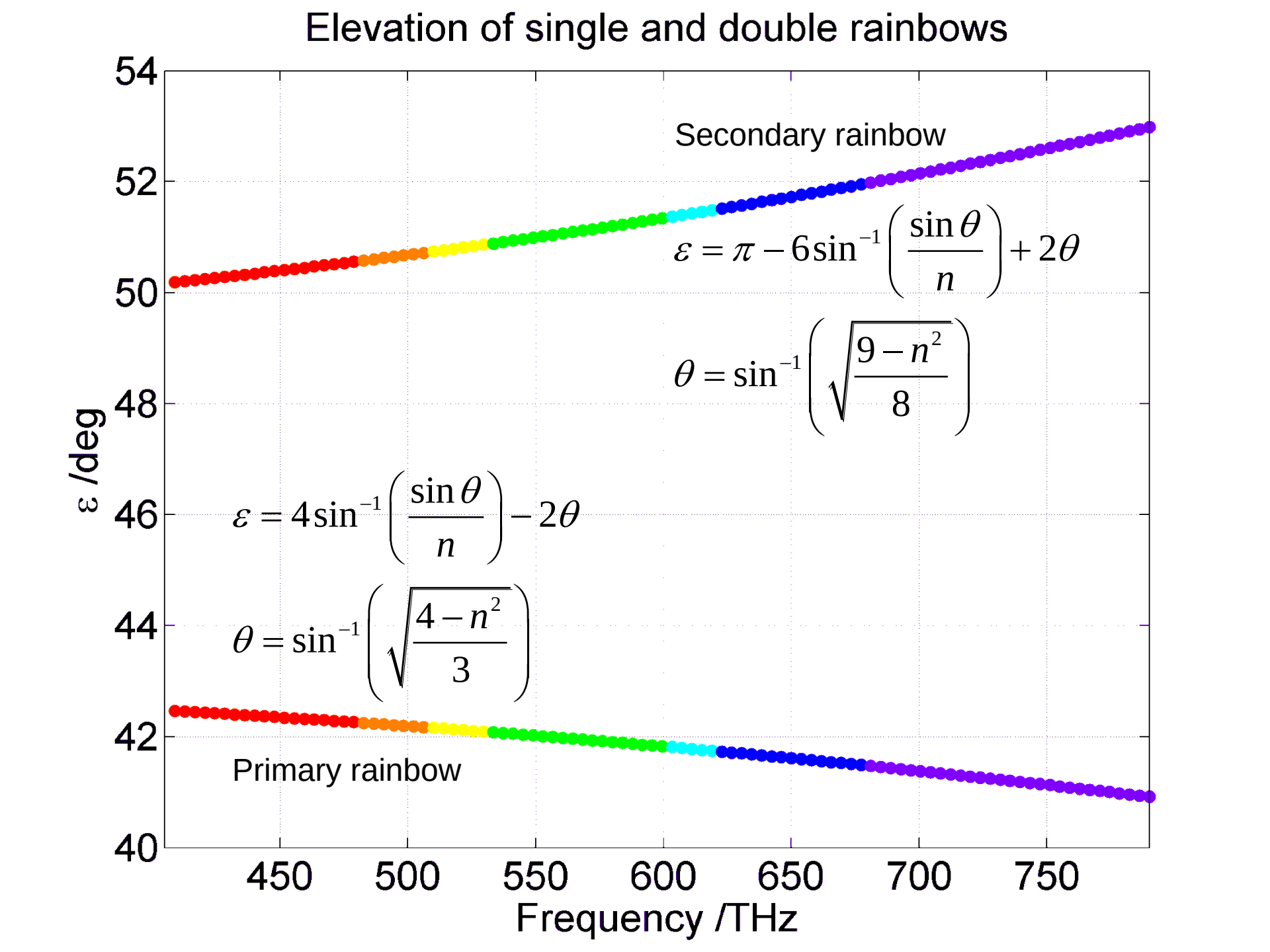
So for light entering a medium of higher refractive index, light is always bent towards the normal to the interface

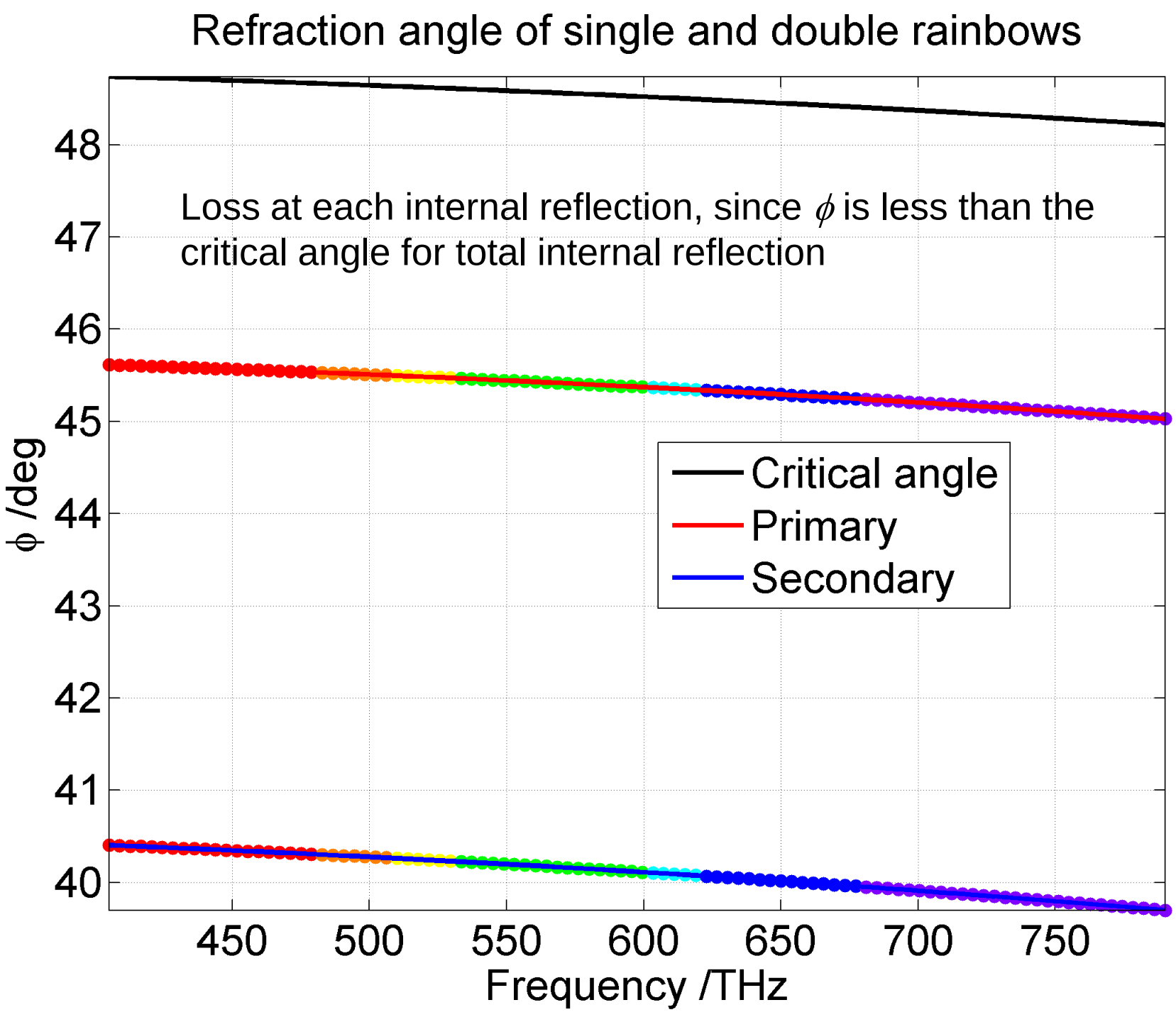

Refractive index of water
Visible range is 405-790THz
Refractive index
Frequency /THz
1.5
1.45
1.4
1.35
1.3
1.25
1.2
1.15
1.1
0
500
1000
1500


y = 1.731 - 0.261x
Proposed empirical equation
between refractive
index and frequency
Once linearized, a line of best fit
can determine
the parameters
$\left(n^2-1\right)^{-2} = 1.731 - 0.261\left(\frac{f}{10^{15}\,\mathrm{Hz}}\right)^2$
y = (n²-1)⁻²
x = (f/10¹⁵Hz)²
1.68
1.66
1.64
1.62
1.6
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55

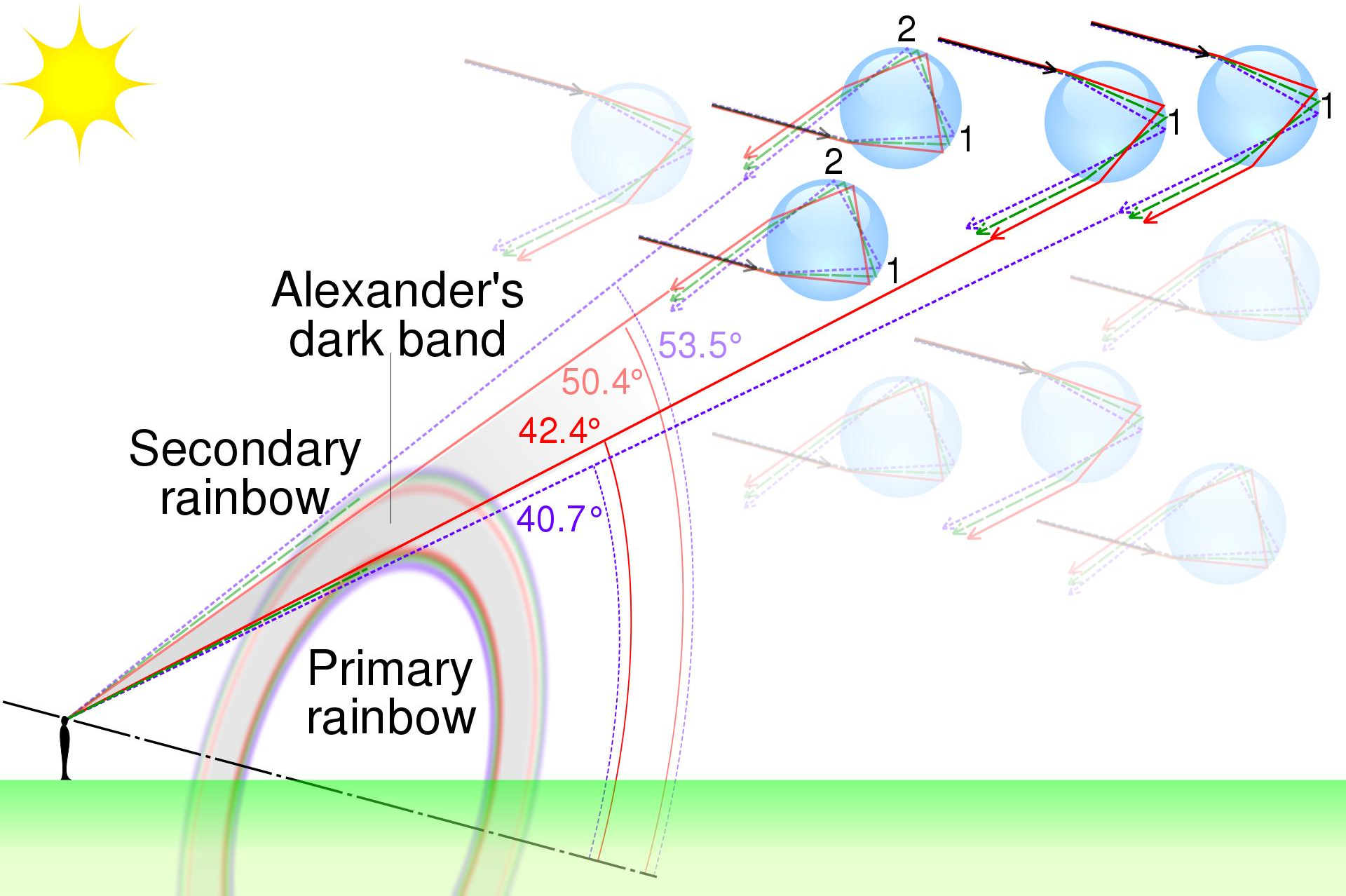

Refractive index of water over visible range 405-790THz
$(n^2-1)^{-2} = 1.731 - 0.261(f/10^{15}\text{Hz})^2$
$n = \sqrt{1+\left(1.731-0.261\left(\frac{f}{10^{15}\,\text{Hz}}\right)^2\right)^{-\frac{1}{2}}}$
Refractive index
1.34
1.338
1.336
1.334
1.332
1.33
450
500
550
600
650
700
750
Frequency /THz
Model
Data


Elevation of deflected beam /deg
Primary ε=40.9° to ,42.5°, Secondary ε=50.2° to 53°
ε /deg
θ /deg
0
20
40
60
80
100
120
140
160
180
$\varepsilon = \pi - 6\sin^{-1}\left(\frac{\sin\theta}{n}\right) + 2\theta$
$\theta = \sin^{-1}\left(\sqrt{\frac{9-n^2}{8}}\right)$
$\varepsilon = 4\sin^{-1}\left(\frac{\sin\theta}{n}\right) - 2\theta$
$\theta = \sin^{-1}\left(\sqrt{\frac{4-n^2}{3}}\right)$


Elevation of single and double rainbows
Secondary rainbow
$\varepsilon = \pi - 6\sin^{-1}\left(\frac{\sin\theta}{n}\right) + 2\theta$
$\theta = \sin^{-1}\left(\sqrt{\frac{9-n^2}{8}}\right)$
$\varepsilon = 4\sin^{-1}\left(\frac{\sin\theta}{n}\right) - 2\theta$
$\theta = \sin^{-1}\left(\sqrt{\frac{4-n^2}{3}}\right)$
Primary rainbow
ε /deg
Frequency /THz
54
52
50
48
46
44
42
40
450
500
550
600
650
700
750


Refraction angle of single and double rainbows
Loss at each internal reflection, since ϕ is less than the critical angle for total internal reflection
Critical angle
Primary
Secondary
ϕ /deg
48
47
46
45
44
43
42
41
40
450
500
550
600
650
700
750
Frequency /THz

https://en.wikipedia.org/wiki/Rainbow



See a *circular* rainbow when flying since rain cloud is illuminated above and below aircraft