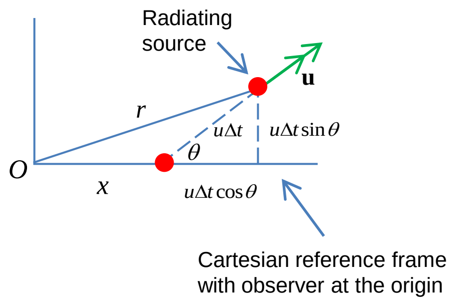
The Doppler Effect

Consider a receding wave source of frequency f. It crosses the x axis of a Cartesian reference frame at angle θ with speed u. The receiver of the waves is stationary at the origin of the Cartesian frame. The speed of waves, relative to the observer, is w. Depending on the velocity of the wave source relative to the observer, the observer will experience a *frequency shift* from f. If the source *recedes*, the frequency *diminishes* and the *wavelength increases* ('redshift'). If the source is *approaching*, the observed *frequency will increase* and the *wavelength will decrease* ('blueshift').

The period T of waves received by an observer (in the x direction) at the frame origin O is:

$$T = \Delta t + \frac{r - x}{w}$$

time between wave crests *at source*; wave speed; extra distance travelled by source between wave crests



From geometry:

$$r = \sqrt{(x + u\Delta t\cos\theta)^2 + u^2\Delta t^2\sin^2\theta}$$

$$r = \sqrt{x^2 + u^2\Delta t^2\cos^2\theta + 2ux\Delta t\cos\theta + u^2\Delta t^2\sin^2\theta}$$

$$r = \sqrt{x^2 + u^2\Delta t^2 + 2ux\Delta t\cos\theta}$$

$$r = x\sqrt{1 + 2\cos\theta\frac{u\Delta t}{x} + \left(\frac{u\Delta t}{x}\right)^2}$$

If $u\Delta t \ll x$ $\quad r \approx x\sqrt{1 + 2\cos\theta\frac{u\Delta t}{x}} \approx x\left(1 + \cos\theta\frac{u\Delta t}{x}\right) = x + u\Delta t\cos\theta$

$$\therefore r - x \approx u\Delta t\cos\theta$$

Hence frequency of radiation received at O is $F = 1/T$ where:

$$\frac{1}{F} = \Delta t + \frac{u\Delta t\cos\theta}{w} = \Delta t\left(1 + \frac{u\cos\theta}{w}\right)$$

In a *Classical* scenario, where u,w are much less than the speed of light: $f = 1/\Delta t$

Hence: $F = \dfrac{f}{1 + \frac{u\cos\theta}{w}} = \dfrac{w}{w + u\cos\theta} f \Rightarrow F = \dfrac{1}{1 + \frac{u}{w}\cos\theta} f$

$\Delta f = F - f \quad \therefore F = \Delta f + f$ ← Define Doppler Shift Δf

$$\therefore \Delta f + f = \frac{1}{1 + \frac{u}{w}\cos\theta} f$$

$$\therefore \Delta f = \left(\frac{1}{1 + \frac{u}{w}\cos\theta} - 1\right) f$$

$$\therefore \Delta f = -\left(\frac{\frac{u}{w}\cos\theta}{1 + \frac{u}{w}\cos\theta}\right) f$$

If the source is *transverse* $\theta = 90^\circ$ and therefore $\Delta f = 0$.

If the source is *on axis* $\theta = 0^\circ$ and therefore:

$$\Delta f = -\frac{u}{w}\left(1 + \frac{u}{w}\right)^{-1} f$$

If the wave speed w is also much greater than the wave source speed u, then we arrive at the most typically quoted Doppler Shift formula:

$$u \ll w$$

$$\Delta f \approx -\frac{u}{w} f$$

In most literature $u = v$ and w is c

Christian Doppler 1803-1853

Define redshift

$$z = \frac{f - F}{F} = \frac{f - \Delta f - f}{\Delta f + f}$$

$$z = -\frac{\Delta f}{\Delta f + f}$$

F observed frequency

f emitted frequency

$$\Delta f + f = \frac{1}{1 + \frac{u}{w}\cos\theta} f$$

$$\therefore \frac{1 + \frac{u}{w}\cos\theta}{f} = \frac{1}{\Delta f + f}$$

$$z = -\frac{\Delta f}{\Delta f + f}$$

$$\therefore z = -\frac{\Delta f}{f}\left(1 + \frac{u}{w}\cos\theta\right)$$

$$\therefore z = \left(\frac{\frac{u}{w}\cos\theta}{1 + \frac{u}{w}\cos\theta}\right)\left(1 + \frac{u}{w}\cos\theta\right)$$

$$\therefore z = \frac{u}{w}\cos\theta$$

Classical redshift formula

Doppler Shift is a highly useful tool in *Cosmology*, since by comparing the *spectra* of light from stars to the light emitted by their constituent gases such as hydrogen or helium *in the laboratory*, we can infer the recession speed of the stars. The *general trend of galaxies to be red-shifted* is the key piece of evidence to suggest the *Big Bang Theory* of the evolution; i.e. that it the **Universe is expanding** and 'began' as an extraordinarily hot and dense 'singularity.'

Classical Doppler Shift formula expressed in terms of wavelength

$$z = \frac{f - F}{F}$$ Redshift

$$\Delta f = F - f$$ Doppler Shift

$$\Delta f = -\left(\frac{\frac{u}{w}\cos\theta}{1 + \frac{u}{w}\cos\theta} \right) f$$ Classical Doppler Shift formula

$$z = \frac{u}{w}\cos\theta$$ Classical redshift formula

$$w = F\lambda_o, \quad w = f\lambda_e$$ ← Wavespeed, frequency and wavelength relationships

$$\therefore \Delta f = F - f = w\left(\frac{1}{\lambda_o} - \frac{1}{\lambda_e} \right)$$

$$\therefore \Delta f = w\left(\frac{\lambda_e - \lambda_o}{\lambda_o \lambda_e} \right) = \left(\frac{\lambda_e - \lambda_o}{\lambda_o} \right) f$$

$$\therefore \frac{\lambda_e - \lambda_o}{\lambda_o} = \frac{\Delta f}{f} = -\frac{\frac{u}{w}\cos\theta}{1 + \frac{u}{w}\cos\theta}$$

$$\therefore \Delta\lambda = \frac{\frac{u}{w}\cos\theta}{1 + \frac{u}{w}\cos\theta}\lambda_o$$

Classical Doppler Shift formula in terms of observed wavelength

$$\Delta\lambda = \lambda_o - \lambda_e$$

$$z = \frac{f - F}{F} = \frac{w\left(\frac{1}{\lambda_e} - \frac{1}{\lambda_o} \right)}{w/\lambda_o} = \frac{\lambda_o}{\lambda_e} - 1$$

$$\therefore 1 + z = \frac{\lambda_o}{\lambda_e}$$

Redshift formula in terms of observed and emitted wavelengths. A **blueshift** is a negative redshift.

$$z = \frac{u}{w}\cos\theta, \quad 1 + z = \frac{\lambda_o}{\lambda_e}$$

$$\therefore \frac{u}{w}\cos\theta = \frac{\lambda_o}{\lambda_e} - 1$$

$$\therefore u\cos\theta = w\left(\frac{\lambda_o - \lambda_e}{\lambda_e} \right)$$

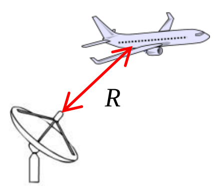
$$\therefore u\cos\theta = \frac{\Delta\lambda}{\lambda_e} w$$

This is probably the most useful formula since it yields the 'radial' velocity component (i.e. in the x direction) in terms of fractional wavelength shift and wave speed

Doppler shift in a radar scenario

RADAR (**RA**dio **D**etection **A**nd **R**anging) is a remote sensing technique where electromagnetic pulses are transmitted and reflected off targets, typically a metallic object like an aircraft. At microwave frequencies (around 1-10GHz) it is possible to detect aircraft at ranges of hundreds of km using a modestly powerful transmitter of the sort that might be installed at an airport.

To measure the range of an aircraft, the radar has a sensitive receiver which listens for the reflection(s) associated with the transmitted pulse. The time delay is the 'there-and-back- time Δt and hence range R can be computed from:



$$R = \tfrac{1}{2} c\Delta t$$

The received signal will be Doppler shifted if the target is moving. Indeed, this fact can be used to design a *Doppler Filter* which will ignore reflections from unmoving 'clutter' such as mountains and other topographic obstacles, and hence enable a small aircraft to be readily identified in a situation where radar echos might otherwise be swamped.

In the radar scenario, we must note an important difference – the 'observed' wavelength was transmitted from the (stationary) radar, reflected off the moving aircraft and then received back by the radar. From the perspective of the aircraft, the radar is a moving source, and therefore the radiation it receives will be Doppler shifted. The radiation it transmits towards the radar will then be Doppler shifted *again*, by the same relative amount. The classical formula is therefore:

$$F_{\text{target}} = \frac{1}{1 + \frac{u}{c}\cos\theta} f$$

Note wave speed w is now the speed of light c

$$F_{\text{radar}} = \frac{1}{1 + \frac{u}{c}\cos\theta} F_{\text{target}}$$

$$\therefore F_{\text{radar}} = \frac{1}{\left(1 + \frac{u}{c}\cos\theta \right)^2} f$$

$$\Delta f = F_{\text{radar}} - f \quad \therefore F_{\text{radar}} = \Delta f + f$$

$$\therefore \Delta f + f = \frac{1}{\left(1 + \frac{u}{c}\cos\theta \right)^2} f$$

$$\therefore \Delta f = \left(\frac{1}{\left(1 + \frac{u}{c}\cos\theta \right)^2} - 1 \right) f$$

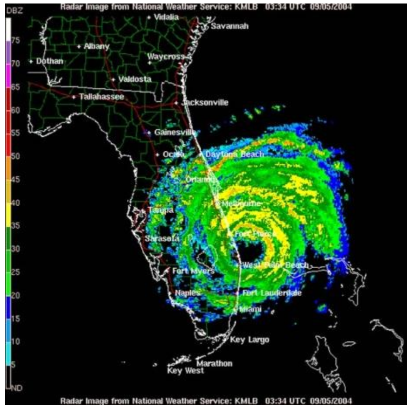
In a Classical scenario $u \ll c$

$$\therefore \Delta f \approx -\frac{2u\cos\theta}{c} f$$

First order binomial expansion

← i.e. in a radar scenario, the Doppler shift is *twice* the value described earlier

Doppler radar can be used to determine **wind speed**. i.e. reflections from moving clouds are processed. This is an important tool in **meteorology**, particularly in forecasting extreme weather events such as hurricanes and tornados



Relativistic Doppler shift

If the wave source speed is a significant fraction of the speed of light, we must take into account effects of *time dilation.*

$$\frac{1}{F} = \Delta t + \frac{u\Delta t\cos\theta}{w} = \Delta t\left(1+\frac{u\cos\theta}{w}\right)$$

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is *still true* if the time Δt between wave crest at source is *measured in the reference frame of the stationary observer.*

$$\gamma = \left(1-\frac{u^2}{c^2}\right)^{-\frac{1}{2}}$$

Using the **generalized Lorentz Transform**

$$\Delta t = \gamma\left(\Delta t' + \frac{\mathbf{u}\cdot\Delta\mathbf{r}'}{c^2}\right)$$

Now since the source is stationary in its frame $\Delta\mathbf{r}' = 0$

Therefore $\Delta t = \gamma\Delta t' = \dfrac{\gamma}{f}$

$$\therefore F = \frac{f}{\gamma\left(1+\frac{u\cos\theta}{w}\right)}$$

$$\Delta f = F - f \quad \therefore F = \Delta f + f$$

$$\therefore \Delta f + f = \frac{1}{\gamma\left(1+\frac{u\cos\theta}{w}\right)} f$$

$$\therefore \Delta f = \left(\frac{1}{\gamma\left(1+\frac{u\cos\theta}{w}\right)} - 1\right) f$$

Relativistic Doppler Shift formula. Note effects of gravity are not included – General Relativity predicts additional redshift* due to the 'spacetime curvature' resulting from the presence of mass (or energy)

$$z = \frac{f-F}{F}$$

$$\therefore 1+z = \frac{f}{F}$$

$$\frac{f}{F} = \gamma\left(1+\frac{u\cos\theta}{w}\right)$$

$$\therefore 1+z = \gamma\left(1+\frac{u\cos\theta}{w}\right)$$

Relativistic redshift formula

Unlike the classical formula, we also get a *transverse Doppler effect* when $\theta = 90^\circ$

$$\frac{\Delta f}{f} = \frac{1}{\gamma} - 1$$

The redshift formula allows us to write the relativistic Doppler shift in terms of wavelength:

$$1+z = \frac{\lambda_o}{\lambda_e}$$

$$\therefore \frac{\lambda_o}{\lambda_e} = \gamma\left(1+\frac{u\cos\theta}{w}\right)$$

For the *special case* of electromagnetic waves propagating on axis towards the observer:

$$\theta = 0^\circ, \ w = c$$

$$\frac{\lambda_o}{\lambda_e} = \left(1-\frac{u^2}{c^2}\right)^{-\frac{1}{2}}\left(1+\frac{u}{c}\right)$$

$$\left(\frac{\lambda_o}{\lambda_e}\right)^2 = \left(1-\frac{u^2}{c^2}\right)^{-1}\left(1+\frac{u}{c}\right)^2$$

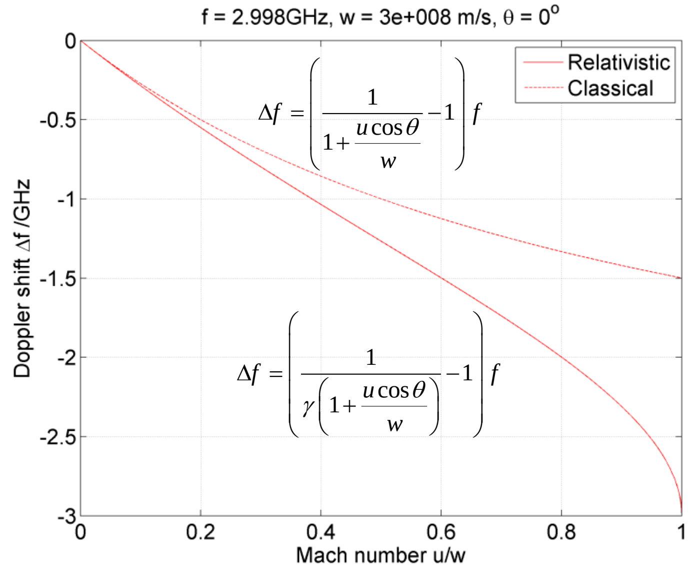
$$\left(\frac{\lambda_o}{\lambda_e}\right)^2 = \left(1+\frac{u}{c}\right)^{-1}\left(1-\frac{u}{c}\right)^{-1}\left(1+\frac{u}{c}\right)^2$$

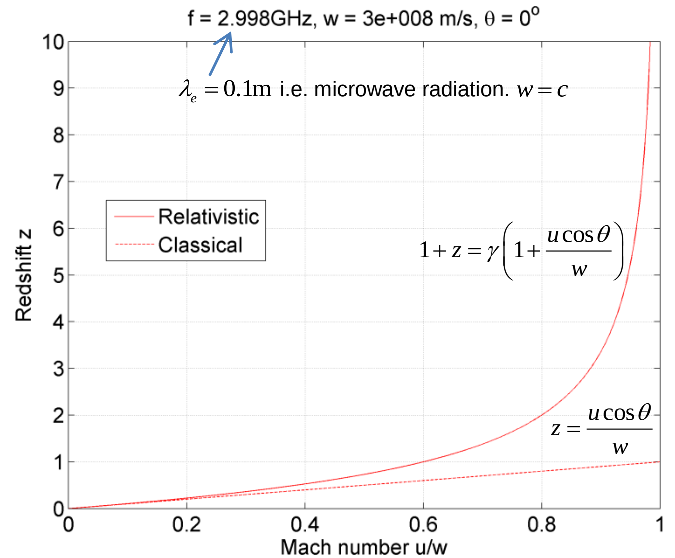
$$\left(\frac{\lambda_o}{\lambda_e}\right)^2 = \left(1-\frac{u}{c}\right)^{-1}\left(1+\frac{u}{c}\right) = \frac{c+u}{c-u}$$

$$(c-u)\left(\frac{\lambda_o}{\lambda_e}\right)^2 - c - u = 0$$

$$u\left(1+\left(\frac{\lambda_o}{\lambda_e}\right)^2\right) = c\left(\left(\frac{\lambda_o}{\lambda_e}\right)^2 - 1\right)$$

$$\therefore \frac{u}{c} = \frac{\lambda_o^2 - \lambda_e^2}{\lambda_o^2 + \lambda_e^2}$$





$$1+z = \frac{1}{\sqrt{1-\frac{2GM}{rc^2}}}$$

$$G = 6.674\times10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$$

***Gravitational redshift** due to a stationary source of light of mass M at distance r from an observer.

Mach's Construction & a graphical visualization of Doppler Shift

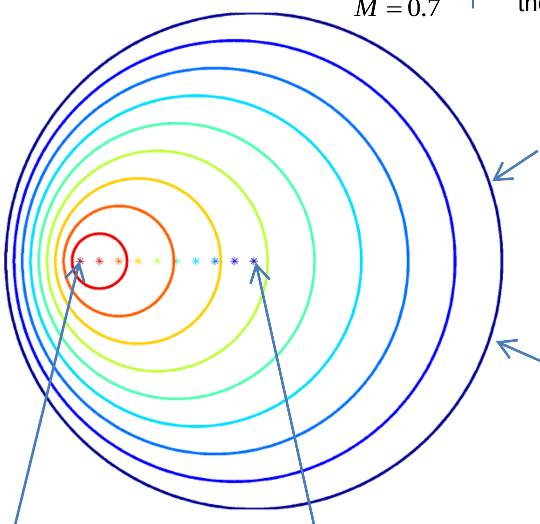
Mach number M is the ratio between the wave source speed and the speed of waves. A Mach number greater than unity implies a 'supersonic' disturbance, which radiate out as a *shock front.*

w is the wave speed
u is the speed of the wave source

$$M = \frac{u}{w}$$

The Doppler shift described previously can be readily visualized for Mach numbers of *less than unity* if one plots a sequence of circular wave fronts being transmitted from the wave source.

$M = 0.7$

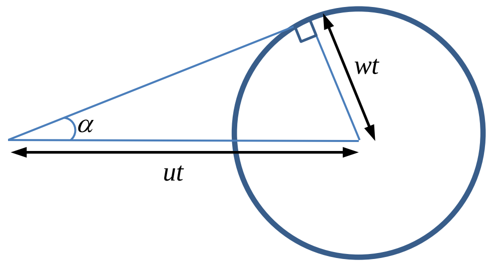


$$1 + z = \frac{\lambda_o}{\lambda_e}$$

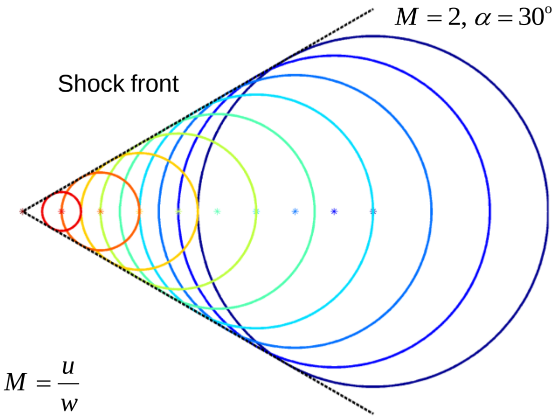
$$\frac{\lambda_o}{\lambda_e} = \gamma\left(1 + \frac{u\cos\theta}{w}\right), \quad \gamma = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}}$$

Summary of redshift and Doppler formulae

For Mach numbers greater than unity a shock front will result, beyond which the 'wave-medium' has no knowledge of the waves produced by the source.



The angle of the shock front (in 3D this will form a 'Mach cone') is the inverse sine of the reciprocal of the Mach number.



$$M = \frac{u}{w}$$

$$ut\sin\alpha = wt$$

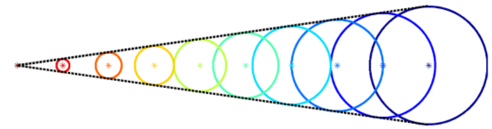
$$\therefore \alpha = \sin^{-1}\tfrac{w}{u}$$

$$\therefore \alpha = \sin^{-1}\tfrac{1}{M}$$

Ernst Mach 1838-1916

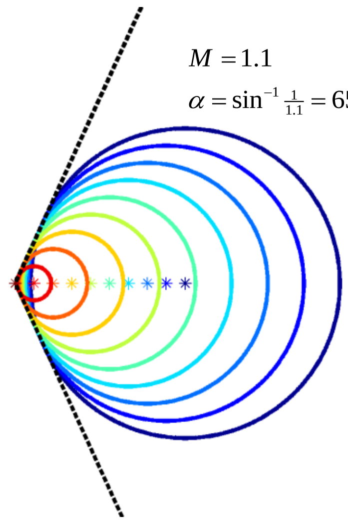
$M = 7$

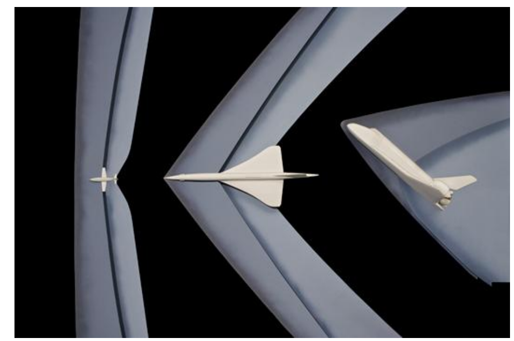
$\alpha = \sin^{-1}\tfrac{1}{7} = 8.2^\circ$



$M = 1.1$

$\alpha = \sin^{-1}\tfrac{1}{1.1} = 65.4^\circ$





Mach cones for various supersonic aircraft (and spacecraft!). Note the cone becomes 'infinitely wide' when $M = 1$. This explains why a sonic boom is heard in all directions when an aircraft breaks the sound barrier. As it gets faster, the cone angle decreases. Hypersonic jets or missiles travelling at Mach 7 would have a cone angle of $\sin^{-1}(1/7) = 8.2^\circ$