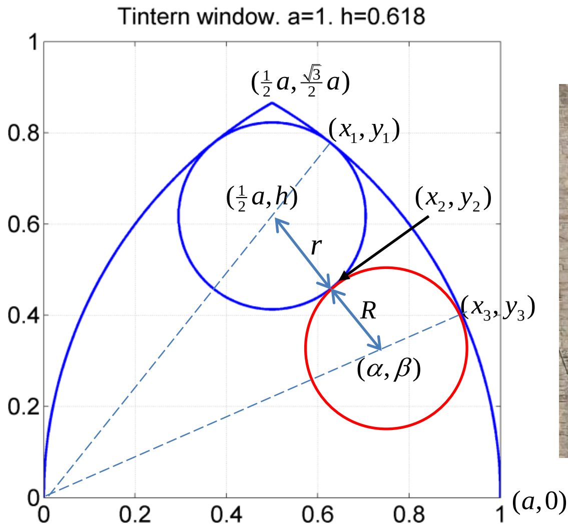
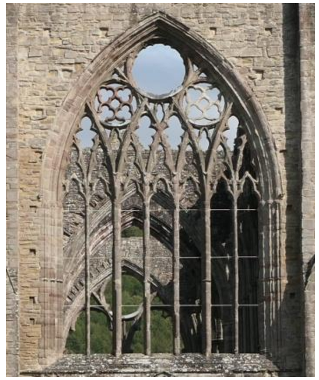




Upper circle: Inputs are $\boxed{a,h}$

$$x_1^2+y_1^2=a^2 \qquad (1)$$

$$\left(x_1-\tfrac12 a\right)^2+\left(y_1-h\right)^2=r^2 \qquad (2)$$

$$\frac{y_1-h}{x_1-\frac12 a}=\frac{h}{\frac12 a} \qquad (3)$$

$$\frac{y_1}{x_1}=\frac{h}{\frac12 a} \qquad (4)$$

$$x_1^2+y_1^2=a^2 \quad (1) \qquad y_1=\frac{2h}{a}x_1 \quad (4)$$

$$\therefore x_1^2\left(1+\frac{4h^2}{a^2}\right)=a^2$$

$$\therefore x_1=\frac{a}{\sqrt{1+\frac{4h^2}{a^2}}}$$

OK to take +ve root

$$\left(x_1-\tfrac12 a\right)^2+\left(y_1-h\right)^2=r^2 \quad (2) \qquad y_1-h=\frac{2h}{a}\left(x_1-\tfrac12 a\right) \quad (3)$$

$$\therefore \left(x_1-\tfrac12 a\right)^2\left(1+\frac{4h^2}{a^2}\right)=r^2$$

$$\therefore r=\sqrt{1+\frac{4h^2}{a^2}}\left(\frac{a}{\sqrt{1+\frac{4h^2}{a^2}}}-\tfrac12 a\right)$$

OK to take +ve root

$$\boxed{\therefore r=a\left(1-\tfrac12\sqrt{1+\frac{4h^2}{a^2}}\right)}$$

Lower circle: Inputs are $\boxed{R}$

$$x_3^2+y_3^2=a^2 \quad (1) \qquad \left(x_3-\alpha\right)^2+\left(y_3-\beta\right)^2=R^2 \quad (2)$$

$$\left(x_2-\alpha\right)^2+\left(y_2-\beta\right)^2=R^2 \quad (3) \qquad \left(x_2-\tfrac12 a\right)^2+\left(y_2-h\right)^2=R^2 \quad (4)$$

$$\frac{y_3-\beta}{x_3-\alpha}=\frac{y_3}{x_3}=\frac{\beta}{\alpha} \quad (5) \qquad \frac{y_2-y_3}{x_3-x_2}=\frac{h-y_2}{x_2-\frac12 a} \quad (6)$$

$$\frac{y_3-\beta}{x_3-\alpha}=\frac{\beta}{\alpha}=\frac{y_3}{x_3} \quad (5) \qquad \left(x_3-\alpha\right)^2+\left(y_3-\beta\right)^2=R^2 \quad (2)$$

$$\therefore \left(x_3-\alpha\right)^2\left(1+\frac{\beta^2}{\alpha^2}\right)=R^2 \Rightarrow x_3=\frac{R}{\sqrt{1+\frac{\beta^2}{\alpha^2}}}+\alpha \Rightarrow x_3=\alpha\left(\frac{R}{\sqrt{\alpha^2+\beta^2}}+1\right)$$

$$\therefore y_3=\frac{\beta}{\alpha}x_3 \Rightarrow y_3=\frac{\beta}{\alpha}\frac{R}{\sqrt{1+\frac{\beta^2}{\alpha^2}}}+\beta \Rightarrow y_3=\beta\left(\frac{R}{\sqrt{\alpha^2+\beta^2}}+1\right)$$

$$x_3=\alpha\left(\frac{R}{\sqrt{\alpha^2+\beta^2}}+1\right) \qquad y_3=\beta\left(\frac{R}{\sqrt{\alpha^2+\beta^2}}+1\right) \qquad x_3^2+y_3^2=a^2 \quad (1)$$

$$\therefore \alpha^2\left(\frac{R}{\sqrt{\alpha^2+\beta^2}}+1\right)^2+\beta^2\left(\frac{R}{\sqrt{\alpha^2+\beta^2}}+1\right)^2=a^2$$

$$\frac{\alpha^2R^2}{\alpha^2+\beta^2}+\alpha^2+\frac{2\alpha^2R}{\sqrt{\alpha^2+\beta^2}}+\frac{\beta^2R^2}{\alpha^2+\beta^2}+\beta^2+\frac{2\beta^2R}{\sqrt{\alpha^2+\beta^2}}=a^2$$

$$R^2+\alpha^2+\beta^2+2\sqrt{\alpha^2+\beta^2}R=a^2 \qquad (7)$$

$$\left(\alpha-\tfrac12 a\right)^2+\left(h-\beta\right)^2=\left(r+R\right)^2 \qquad (8)$$

$$\therefore \alpha=\tfrac12 a+\sqrt{\left(r+R\right)^2-\left(h-\beta\right)^2}$$

$$\left(\alpha-\tfrac12 a\right)^2+\left(h-\beta\right)^2=\left(r+R\right)^2 \qquad (8)$$

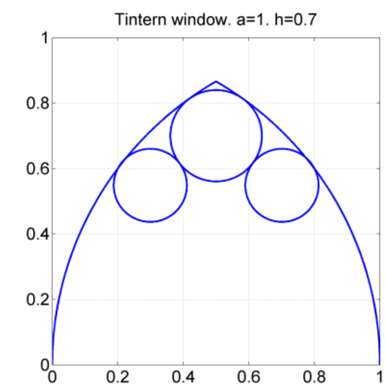
$$\therefore \alpha=\tfrac12 a+\sqrt{\left(r+R\right)^2-\left(h-\beta\right)^2}$$

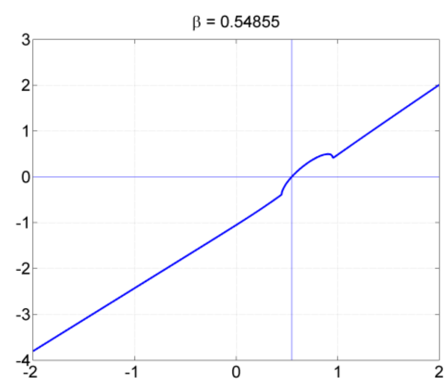
$$R^2+\alpha^2+\beta^2+2\sqrt{\alpha^2+\beta^2}R=a^2$$

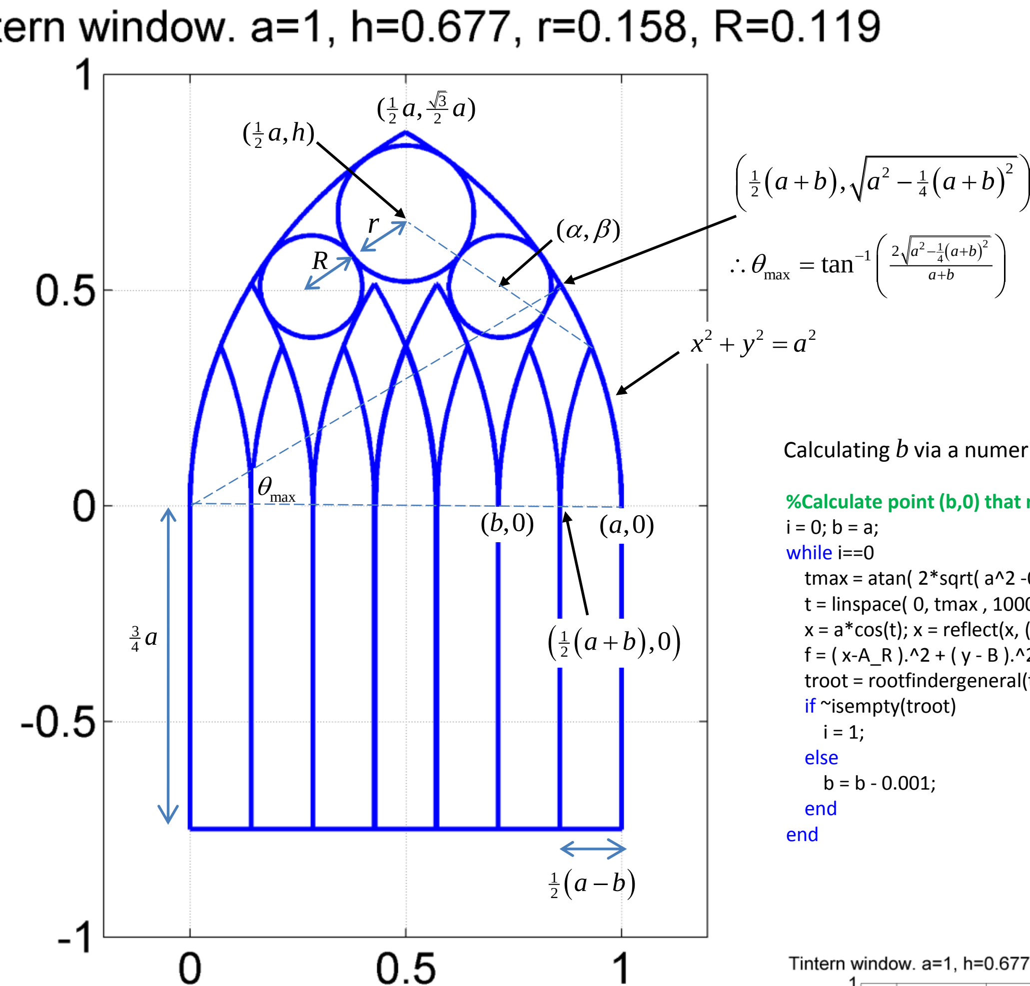
$$\therefore R^2+\left(\tfrac12 a+\sqrt{\left(r+R\right)^2-\left(h-\beta\right)^2}\right)^2+\beta^2+\ldots$$

$$\boxed{2R\sqrt{\left(\tfrac12 a+\sqrt{\left(r+R\right)^2-\left(h-\beta\right)^2}\right)^2+\beta^2}-a^2=0}$$

Solve for β via a numeric root-finding method







Idea is to run the whole process in a loop, varying h until the optimum b is close as possible to $5a/7$.

Note:

$b = \frac{5}{7}a$

$\therefore \theta_{max} = \tan^{-1}\left(\frac{2\sqrt{a^2 - \frac{1}{4}a^2\frac{144}{49}}}{\frac{12}{7}a}\right)$

$\therefore \theta_{max} = \tan^{-1}\left(\frac{7\sqrt{\frac{13}{49}}}{6}\right)$

$\therefore \theta_{max} = \tan^{-1}\left(\frac{\sqrt{13}}{6}\right) \approx 31^\circ$

Arch arc coordinates

$\therefore \theta_{max} = \tan^{-1}\left(\frac{2\sqrt{a^2-\frac{1}{4}(a+b)^2}}{a+b}\right)$

$x = a\cos\theta,\ y = a\sin\theta,\ 0 \le \theta \le \theta_{max}$

Then reflect arc in $x = \frac{1}{2}(a+b)$

Start with $b = a$, and then decrement by 0.001. Determine b such that:

$$f(\theta) = (x(\theta) - \alpha)^2 + (y(\theta) - \beta)^2 - R^2$$

passes through zero for the first time.
i.e. the left (reflected) arc is a tangent to the lower circle.

Calculating b via a numerical method

```
%Calculate point (b,0) that right arch intersects x axis
i = 0; b = a;
while i==0
   tmax = atan( 2*sqrt( a^2 -0.25*(a+b)^2 )/(a+b) );
   t = linspace( 0, tmax , 1000 );
   x = a*cos(t); x = reflect(x, (a+b)/2 ); y = a*sin(t);
   f = ( x-A_R ).^2 + ( y - B ).^2 - R^2;
   troot = rootfindergeneral(t,f);
   if ~isempty(troot)
      i = 1;
   else
      b = b - 0.001;
   end
end
```

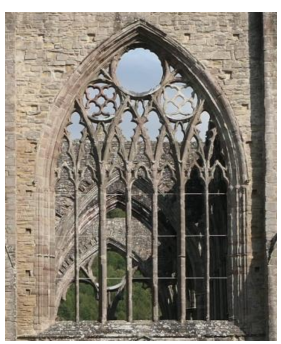
Once b is found, translate the arches.
Repeat the process for the smaller arches.

For the Tintern window

$R = \frac{3}{4}r$

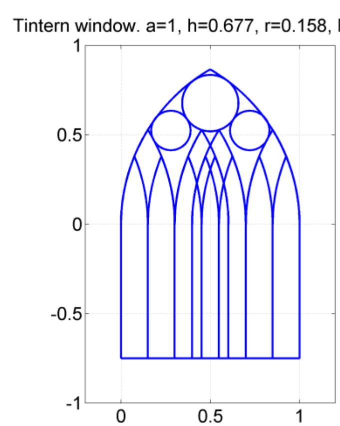
$\frac{1}{2}(a-b) = \frac{1}{7}a$

This is AMAZING why this works!



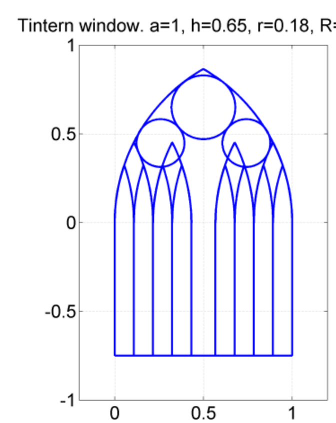
$\frac{1}{2}(a-b) = \frac{1}{7}a$

$\therefore b = \frac{5}{7}a \approx 0.7143a$



$R = 0.7r$

Arches don't overlap properly

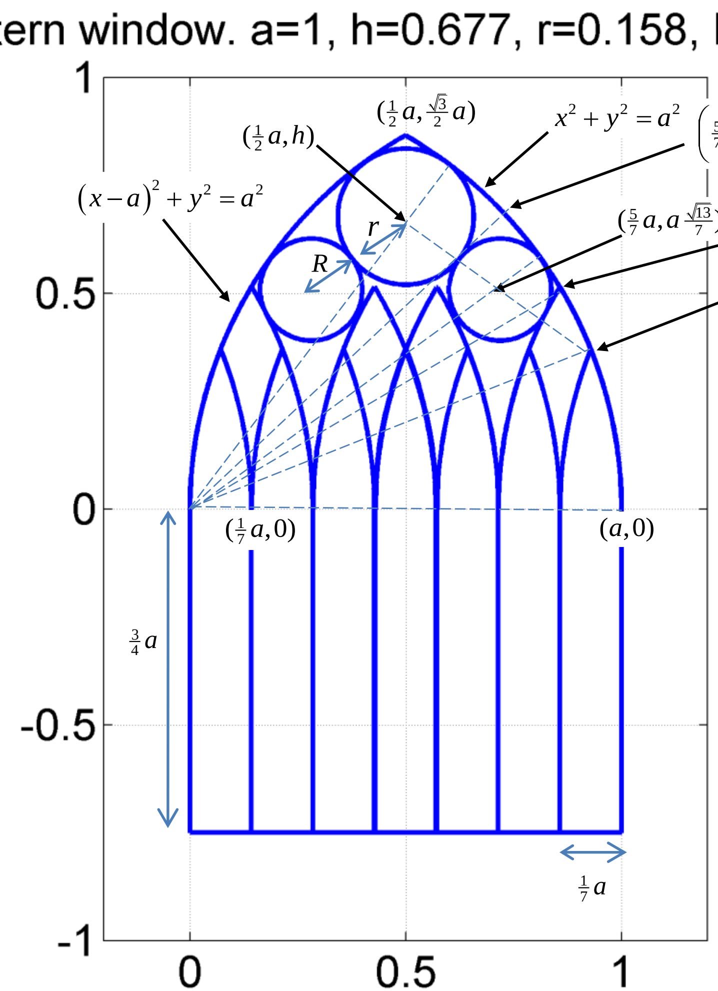


$R = \frac{3}{4}r$

$h = 0.65a$

Reflected arches are no longer tangents and we don't get seven vertical windows

Alternatively, start with the actual Tintern window and work upwards! Can we calculate h analytically? (Nearly ...)



i.e. it looks like it is tangent to the left outer arch. Exercise for the keen!

Calculate radius of small tangent circles

$$x^2+y^2=a^2$$

$$\left(x-\tfrac{5}{7}a\right)^2+\left(y-\tfrac{\sqrt{13}}{7}a\right)^2=R^2$$

$$y=\tfrac{\sqrt{13}}{5}x$$

$$\therefore x^2\left(1+\tfrac{13}{25}\right)=a^2 \Rightarrow x=\tfrac{5}{\sqrt{38}}a \therefore y=\tfrac{\sqrt{494}}{38}a$$

$$\therefore R=a\sqrt{\left(\tfrac{5}{\sqrt{38}}-\tfrac{5}{7}\right)^2+\left(\tfrac{\sqrt{494}}{38}-\tfrac{\sqrt{13}}{7}\right)^2}\approx 0.1194a$$

Guess centre coordinates (!) So now find R such that small circle is tangent to outer arch.

Calculate radius and centre of top circle

Pythagoras:

$$(r+R)^2=\left(\tfrac{5}{7}a-\tfrac{1}{2}a\right)^2+\left(h-a\tfrac{\sqrt{13}}{7}\right)^2$$

Intersection of outer arch and top circle

$$x^2+y^2=a^2$$

$$\left(x-\tfrac{1}{2}a\right)^2+(y-h)^2=r^2$$

Intersection point on same line from (0,0)

$$\frac{h}{\frac{1}{2}a}=\frac{y}{x}=\frac{y-h}{x-\frac{1}{2}a}$$

$$y-h=\left(x-\tfrac{1}{2}a\right)\tfrac{2h}{a}$$

$$\therefore \left(x-\tfrac{1}{2}a\right)^2\left(1+\tfrac{4h^2}{a^2}\right)=r^2$$

$$\therefore x=\frac{r}{\sqrt{1+\frac{4h^2}{a^2}}}+\tfrac{1}{2}a \quad \therefore y=\tfrac{2h}{a}x$$

$$x^2+y^2=a^2$$

$$\therefore \left(\frac{r}{\sqrt{1+\frac{4h^2}{a^2}}}+\tfrac{1}{2}a\right)^2\left(1+\tfrac{4h^2}{a^2}\right)=a^2$$

$$\therefore \frac{r}{\sqrt{1+\frac{4h^2}{a^2}}}+\tfrac{1}{2}a=\frac{a}{\sqrt{1+\frac{4h^2}{a^2}}}$$

$$\therefore r=a-\tfrac{1}{2}\sqrt{1+\tfrac{4h^2}{a^2}}\,a$$

$$(r+R)^2=\left(\tfrac{5}{7}a-\tfrac{1}{2}a\right)^2+\left(h-a\tfrac{\sqrt{13}}{7}\right)^2$$

Can't solve easily analytically

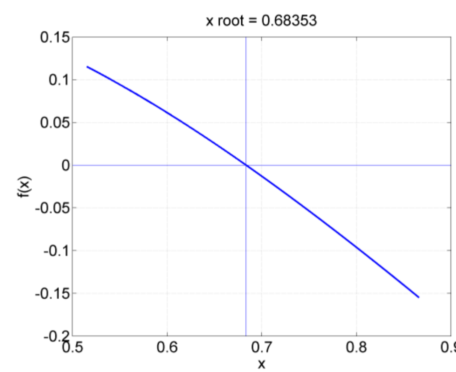
$$\therefore \left(a-\tfrac{1}{2}\sqrt{1+\tfrac{4h^2}{a^2}}\,a+R\right)^2=\tfrac{9}{196}a^2+\left(h-a\tfrac{\sqrt{13}}{7}\right)^2$$

$$f(h)=\left(a-\tfrac{1}{2}\sqrt{1+\tfrac{4h^2}{a^2}}\,a+R\right)^2-\tfrac{9}{196}a^2-\left(h-a\tfrac{\sqrt{13}}{7}\right)^2$$

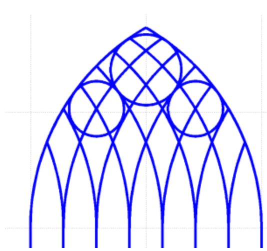
$$f(h)=0 \Rightarrow h\approx 0.683526a$$

$$\therefore r=0.1531186a$$

Note: $\frac{r}{R}=1.28273, \quad \frac{4}{3}=1.333$



Plot white patches for circles to cover up the arches underneath



```
%Plot all arches
for n=1:7
   w = a - (n-1)*a/7; x0 = (n-1)*a/7;
   arch_plotter( a, w, x0, L, [0,0,1] );
   arch_plotter( a, w, 0, L, [0,0,1] );
end
```

```
%Arch plotter. Plots an arch formed from an arc of a circle of radius a
%and arch width w, starting at x=x0. L is the linewidth, RGB is the line
%colour
function arch_plotter( a,w,x0,L,RGB )
A = a - w/2; B = sqrt( a^2 - A^2 );
t = linspace( 0, atan(B/A), 1000) ; d = a-w-x0;
xR = a*cos(t); xL = 2*A-  xR;  xR = xR - d; xL = xL - d; y = a*sin(t);
plot( xR,y, 'color',RGB,'linewidth',L );
plot( xL,y, 'color',RGB,'linewidth',L );
```