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www.eclecticon.info (Maths)

www.wolframalpha.com

TRIGNOMETRICA

A. French. May 2013

$\boxed{\pi \text{ radians} = 180^\circ}$

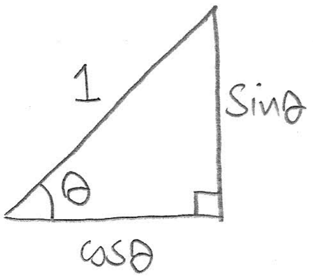
radians must be used for calculus involving trigonometry

Fundamental definitions

$$\frac{1}{\sin\theta} = \operatorname{cosec}\theta$$

$$\frac{1}{\cos\theta} = \sec\theta$$

$$\frac{1}{\tan\theta} = \cot\theta$$



$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

{ From this figure, $0 \leq \theta \leq 180^\circ$ }

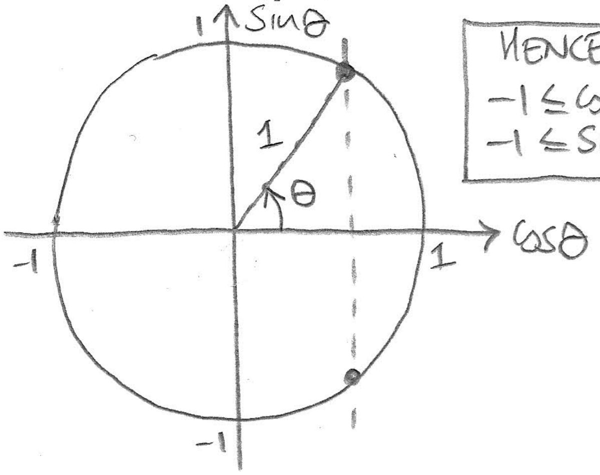
Pythagoras' theorem

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

$\div \sin^2\theta$ → $\boxed{1 + \cot^2\theta = \operatorname{cosec}^2\theta}$

$\div \cos^2\theta$ → $\boxed{\tan^2\theta + 1 = \sec^2\theta}$

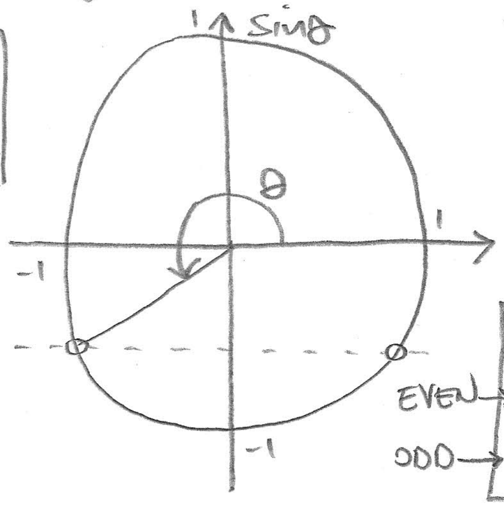
Extend definition of $\sin\theta$ and $\cos\theta$ noting $x = \cos\theta$, $y = \sin\theta$ means $x^2 + y^2 = 1$ ie equation of the $\boxed{\text{unit circle}}$



HENCE

$-1 \leq \cos\theta \leq 1$

$-1 \leq \sin\theta \leq 1$



Convention

[MEASURE θ anticlockwise from 'x' axis]

NOTE

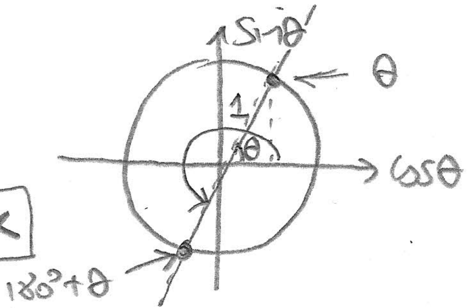
EVEN → $\cos(-\theta) = \cos\theta$

ODD → $\sin(-\theta) = -\sin\theta$

Geometric description of $\cos\theta = k$ (eg $k = 0.6$) ie two solutions in range $0 \leq \theta \leq 360^\circ$

Geometric description of $\sin\theta = k$ (eg $k = -0.6$) i.e two solutions in range $0 \leq \theta \leq 360^\circ$

Since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\tan\theta = k$ corresponds to a diameter of the unit circle of $\boxed{\text{gradient } k}$ so θ repeats every 180°.



$\sin\theta$, $\cos\theta$, $\tan\theta$ also have characteristic curves. $\sin\theta$ and $\cos\theta$ form the basis of wave-like or oscillatory functions. Much of physics uses wave functions of time (t) or space (x) to describe the physical world

$$\boxed{\psi(x,t) = A\cos(kx - 2\pi f t)}$$

Amplitude   Wavenumber   frequency

$$\boxed{k = \frac{2\pi}{\lambda}}$$

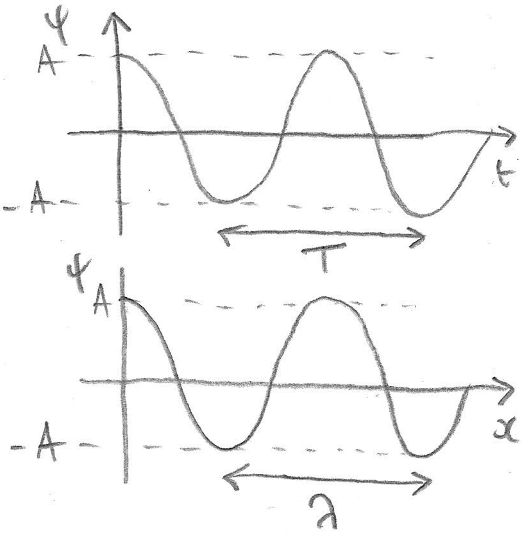
wave speed is $\boxed{c = f\lambda}$

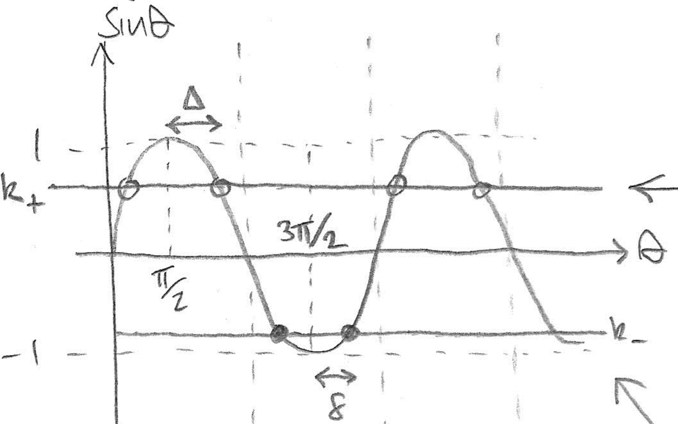
λ is the wavelength

T is the period

If $\boxed{\psi = A\cos\theta}$

θ is called the <u>phase</u> of the wave, i.e. where you are in the cycle.



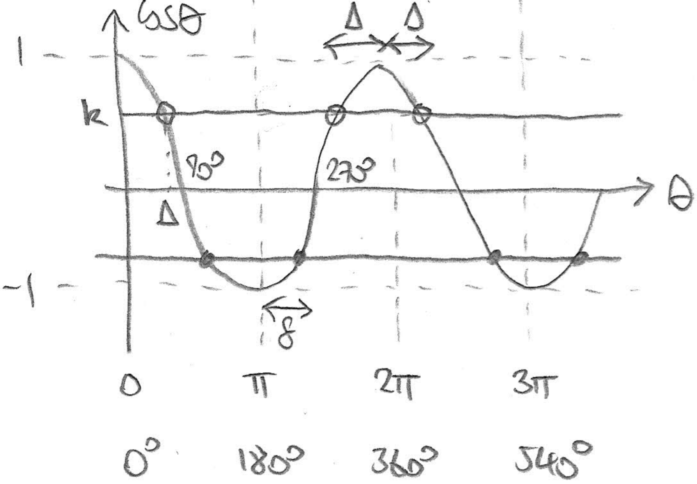


Solutions to $\sin\theta = k$ $(0 < k < 1)$

These are $\boxed{\theta = \frac{\pi}{2} \pm \Delta + 2\pi z}$

where z is an integer.

If $k < 0$, $k > -1$ they would be $\boxed{\theta = \frac{3\pi}{2} \pm \delta + 2\pi z}$



For $0 < k < 1$ and $\cos\theta = k$

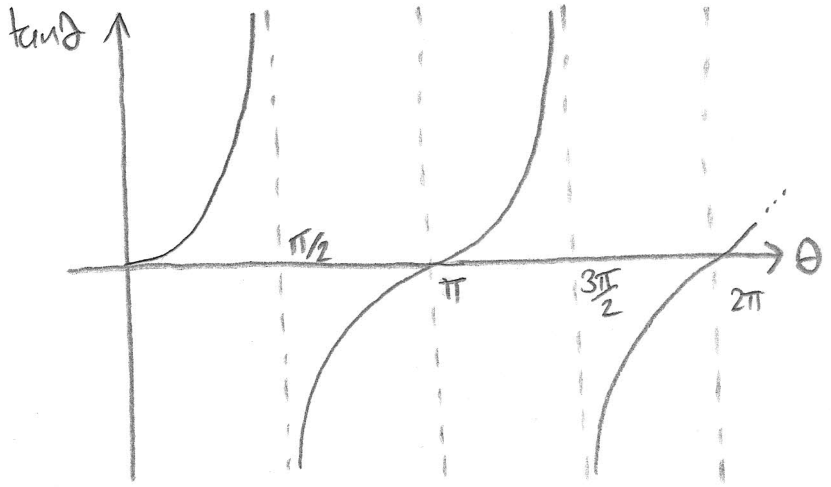
Solutions are $\boxed{\pm\Delta + 2\pi z = \theta}$

and for $k > -1$, $k < 0$

$$\boxed{\theta = \pi \pm \delta + 2\pi z}$$

Find Δ, δ by working out $\cos^{-1}k$ (or $\sin^{-1}k$) via <u>exact</u> (eg $\sin 30° = \frac{1}{2}$) or calculator means and the subtracting this from π or $\frac{\pi}{2}$ as appropriate.

NOTE
$$\boxed{\begin{aligned} \pm\cos\theta &= \sin(\theta \pm \tfrac{\pi}{2}) \\ \mp\sin\theta &= \cos(\theta \pm \tfrac{\pi}{2}) \end{aligned}}$$



$\tan\theta$ is periodic by π radians or $180°$

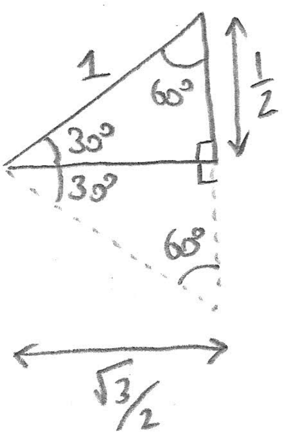
ie $\boxed{\tan(\theta + z\pi) = \tan\theta}$

where z is integer

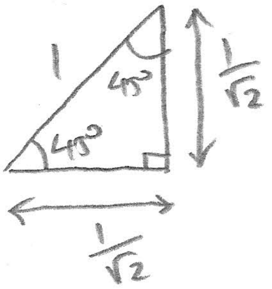
$\tan\frac{\pi}{2} = \pm\infty$

(Think  and steeper!

Special triangles (Exact results)



$\cos 30°$	$=$	$\sqrt{3}/2$
$\sin 30°$	$=$	$1/2$
$\tan 30°$	$=$	$1/\sqrt{3}$
$\cos 60°$	$=$	$\frac{1}{2}$
$\sin 60°$	$=$	$\sqrt{3}/2$
$\tan 60°$	$=$	$\sqrt{3}$



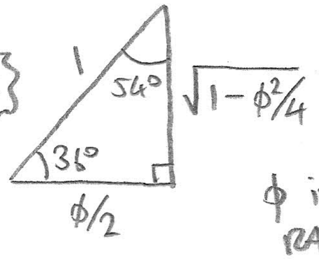
$\sin 45° = \frac{1}{\sqrt{2}}$

$\cos 45° = \frac{1}{\sqrt{2}}$

$\tan 45° = 1$

Also:

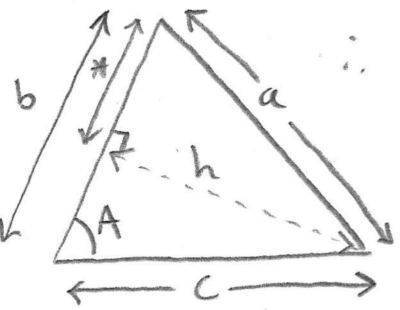
Prove via regular pentagon construction!



ϕ is the GOLDEN RATIO

$\phi = \frac{1+\sqrt{5}}{2}$

Cosine rule



$a^2 = *^2 + h^2$

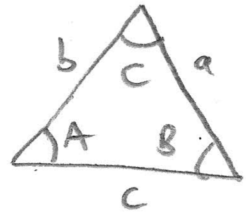
$\therefore\ a^2 = (b - c\cos A)^2 + h^2 \quad (1)$

Also: $c^2 = (c\cos A)^2 + h^2 \quad (2)$

$(2) - (1)$: $c^2 - a^2 = c^2\cos^2 A - \{b^2 - 2bc\cos A + c^2\cos^2 A\}$

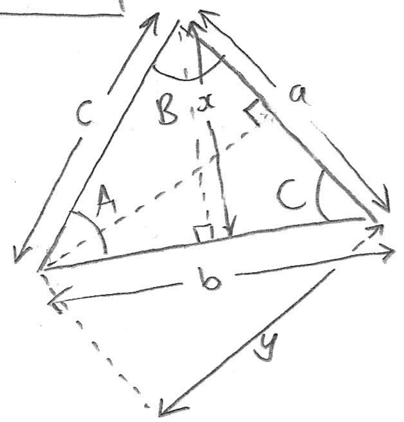
$\therefore\ \boxed{c^2 + b^2 - 2bc\cos A = a^2}$

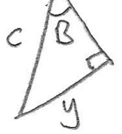
$* = b - c\cos A$



Hence $\boxed{\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos B \\ c^2 &= a^2 + b^2 - 2ab\cos C \end{aligned}}$

Sine rule



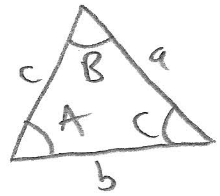


$$\left.\begin{aligned} y &= c\sin B \\ y &= b\sin C \end{aligned}\right\} \therefore \; c\sin B = b\sin C \;\Rightarrow\; \boxed{\frac{\sin B}{b} = \frac{\sin C}{c}}$$

Also

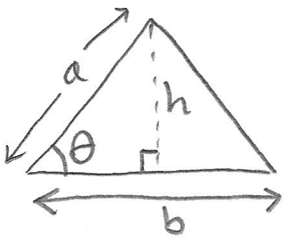
$$\left.\begin{aligned} c\sin A &= x \\ a\sin C &= x \end{aligned}\right\} \therefore \; c\sin A = a\sin C \;\Rightarrow\; \boxed{\frac{\sin A}{a} = \frac{\sin C}{c}}$$

Hence



$$\boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}}$$

Area of a triangle

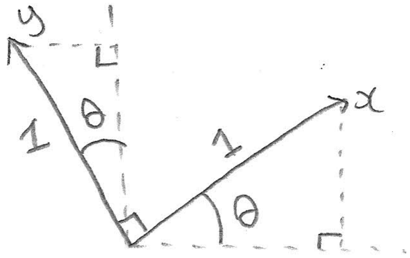


$$\text{Area} = \frac{1}{2}bh = \boxed{\frac{1}{2}ab\sin\theta}$$

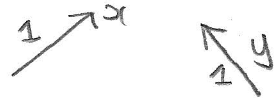
Rotation Matrix

$\underline{\underline{R}}$

(anticlockwise about the origin by θ)



$$\underline{\underline{R}} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$



Now $\boxed{\underline{\underline{R}}_{A\pm B} = \underline{\underline{R}}_{\pm B}\,\underline{\underline{R}}_{A}}$ ie a rotation by A followed by a rotation by $\pm B$.

Hence

$$\begin{pmatrix} \cos(A\pm B) & -\sin(A\pm B) \\ \sin(A\pm B) & \cos(A\pm B) \end{pmatrix} = \begin{pmatrix} \cos(\pm B) & -\sin(\pm B) \\ \sin(\pm B) & \cos(\pm B) \end{pmatrix}\begin{pmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{pmatrix}$$

using $\boxed{\cos(\pm B) = \cos B}$ and $\boxed{\sin(\pm B) = \pm\sin B}$

$$\Rightarrow \boxed{\begin{aligned} \cos(A\pm B) &= \cos A\cos B \mp \sin A\sin B \\ \sin(A\pm B) &= \pm\sin B\cos A + \cos B\sin A = \sin A\cos B \pm \sin B\cos A \end{aligned}}$$

SINE AND COSINE ADDITION FORMULAE

$\boxed{\text{Double angle formulae}}$ eg $\sin(A+A)$

$$\sin 2A = 2\sin A\cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

$\boxed{\text{Triple angle formulae}}$ eg $\cos(2A+A)$

$$\sin 3A = 3\sin A - 4\sin^3 A$$
$$\cos 3A = 4\cos^3 A - 3\cos A$$

$\boxed{\text{Expansions of trig powers}}$

$$\sin^2 A = \cos^2 A - \cos 2A$$
$$= 1 - \sin^2 A - \cos 2A$$
$$\therefore 2\sin^2 A = 1 - \cos 2A$$
$$\therefore \boxed{\sin^2 A = \frac{1-\cos 2A}{2}}$$

Similarly: $\boxed{\cos^2 A = \frac{1}{2}(1+\cos 2A)}$

$$\boxed{\begin{aligned}\cos^3 A &= \tfrac{1}{4}(3\cos A + \cos 3A)\\ \sin^3 A &= \tfrac{1}{4}(3\sin A - \sin 3A)\end{aligned}}$$

$$\boxed{\begin{aligned}\cos(A\pm B) &= \cos A\cos B \mp \sin A\sin B\\ \sin(A\pm B) &= \sin A\cos B \pm \cos A\sin B\end{aligned}}$$

Aide memoir :)

$$*(A\pm B) = *(A)\cos B + \underbrace{*'(A)}_{\text{First derivative}}\sin B$$

$*$: cos or sin

$$\boxed{\begin{aligned}\frac{d}{d\theta}\sin\theta &= \cos\theta\\ \frac{d}{d\theta}\cos\theta &= -\sin\theta\end{aligned}}$$

$$\tan(A\pm B) = \frac{\sin(A\pm B)}{\cos(A\pm B)}$$
$$= \frac{\sin A\cos B \pm \cos A\sin B}{\cos A\cos B \mp \sin A\sin B}$$
$$= \frac{\cos A\cos B(\tan A \pm \tan B)}{\cos A\cos B(1 \mp \tan A\tan B)}$$
$$\therefore \boxed{\tan(A\pm B) = \frac{\tan A \pm \tan B}{1\mp \tan A\tan B}}$$

Can rewrite as

$$\boxed{\begin{aligned}R\cos(A\pm B) &= a\cos B \mp b\sin B\\ R\sin(A\pm B) &= b\cos B \pm a\sin B\\ R &= \sqrt{a^2+b^2} \qquad a = R\cos A\\ & \qquad\qquad\qquad b = R\sin A\end{aligned}}$$

INVERSES.

$$\cos^{-1}A \pm \cos^{-1}B = \cos^{-1}\left(AB \mp \sqrt{(1-A^2)(1-B^2)}\right)$$
$$\sin^{-1}A \pm \sin^{-1}B = \sin^{-1}\left(A(1-B^2)^{\frac{1}{2}} \pm B(1-A^2)^{\frac{1}{2}}\right)$$
$$\tan^{-1}A \pm \tan^{-1}B = \tan^{-1}\left(\frac{A\pm B}{1\mp AB}\right)$$

Note $\boxed{\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x}$

$\boxed{\text{Complex numbers}}$ De Moivre's Theorem $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ *

$\boxed{i^2 = -1}$

$\therefore \left(e^{i\theta}\right)^n = e^{in\theta} = \cos n\theta + i\sin n\theta$

Hence $\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$

$$\boxed{\begin{aligned} \sin\theta &= \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) \\ \cos\theta &= \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \end{aligned}}$$

This result (+ the binomial theorem) allows us to derive expansions of $\cos^n\theta$ and $\sin^n\theta$ in terms of lower powers of n

$\cos n\theta + i\sin n\theta$

$$= \binom{n}{0}\cos^n\theta + \binom{n}{1}\cos^{n-1}\theta\,(i\sin\theta) + \binom{n}{2}\cos^{n-2}\theta\,(i\sin\theta)^2 + \dots$$
$$+ \dots \binom{n}{n}(i\sin\theta)^n$$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array} \leftarrow \text{Pascal's triangle}$$

Example: let $\boxed{n = 3}$

$$\therefore \cos 3\theta + i\sin 3\theta = \cos^3\theta + 3\cos^2\theta\,(i\sin\theta) + 3\cos\theta\,(i\sin\theta)^2 + (i\sin\theta)^3$$
$$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

$\therefore$ (real terms): $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$

$$\Rightarrow \cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\therefore \boxed{\cos^3\theta = \frac{\cos 3\theta + 3\cos\theta}{4}}$$

(terms in i): $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$
$$= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta$$
$$= 3\sin\theta - 4\sin^3\theta$$

$$\therefore \boxed{\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}}$$

* PROOF OF DE MOIVRE'S THEOREM

let $z = \cos\theta + i\sin\theta$

$\therefore \frac{dz}{d\theta} = -\sin\theta + i\cos\theta$

$\Rightarrow \frac{dz}{d\theta} = iz$

$\therefore \int \frac{dz}{z} = i\int d\theta$

$\Rightarrow \ln z = i\theta$

$\therefore \boxed{z = e^{i\theta}}$

Use this result recursively to find $\sin^4\theta$, $\sin^5\theta$ etc....

$$\sin^2 x = \tfrac{1}{2}(1-\cos 2x)$$
$$\sin^3 x = \tfrac{1}{4}(3\sin x - \sin 3x)$$

{From Wolfram Alpha}

$$\sin^4 x = \tfrac{1}{8}(3 - 4\cos 2x + \cos 4x)$$
$$\sin^5 x = \tfrac{1}{16}(10\sin x - 5\sin 3x + \sin 5x)$$
$$\sin^6 x = \tfrac{1}{32}(10 - 15\cos 2x + 6\cos 4x - \cos 6x)$$
$$\sin^7 x = \tfrac{1}{64}(35\sin x - 21\sin 3x + 7\sin 5x - \sin 7x)$$
$$\sin^8 x = \tfrac{1}{128}(35 - 56\cos 2x + 28\cos 4x - 8\cos 6x + \cos 8x)$$
$$\sin^9 x = \tfrac{1}{256}(126\sin x - 84\sin 3x + 36\sin 5x - 9\sin 7x + \sin 9x)$$

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$
$$\cos^3 x = \tfrac{1}{4}(3\cos x + \cos 3x)$$
$$\cos^4 x = \tfrac{1}{8}(3 + 4\cos 2x + \cos 4x)$$
$$\cos^5 x = \tfrac{1}{16}(10\cos x + 5\cos 3x + \cos 5x)$$
$$\cos^6 x = \tfrac{1}{32}(10 + 15\cos 2x + 6\cos 4x + \cos 6x)$$
$$\cos^7 x = \tfrac{1}{64}(35\cos x + 21\cos 3x + 7\cos 5x + \cos 7x)$$
$$\cos^8 x = \tfrac{1}{128}(35 + 56\cos 2x + 28\cos 4x + 8\cos 6x + \cos 8x)$$
$$\cos^9 x = \tfrac{1}{256}(126\cos x + 84\cos 3x + 36\cos 5x + 9\cos 7x + \cos 9x)$$

The above results enable integrals of the form $\int \cos^n x\,dx$ or $\int \sin^n x\,dx$ to be found exactly. (n is integer).

$\boxed{\text{Question:}}$ what is the expansion of $T^n(x)$ in terms of $\sin(px)$ and $\cos(qx)$ where $T = \sin x$ or $T = \cos x$ and n, p, q are integers.

Is there a formula that gives the coefficients?

ie could you use a computer to work out the exact solution to $\int_0^{\pi/4} \sin^{100} x\,dx$?

Calculus results involving trigonometric functions

If x is in radians

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

Basic results

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

Since $\int \ldots dx$ inverse of $\frac{d}{dx}$

Using the quotient rule

$$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x \qquad \therefore \boxed{\frac{d}{dx}\tan x = \sec^2 x}$$

These can be generalized using the chain rule $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$

$$\frac{d}{dx}\sin ax = a\cos ax$$

$$\frac{d}{dx}\cos ax = -a\sin ax$$

$$\frac{d}{dx}\tan ax = a.\sec^2 ax$$

a is a constant

Other useful results

$$\frac{d}{dx}\sec ax = (a\sec ax)\tan ax$$

$$\frac{d}{dx}\sin^{-1} ax = \frac{a}{\sqrt{1-a^2x^2}}$$

$$\frac{d}{dx}\cos^{-1} ax = -\frac{a}{\sqrt{1-a^2x^2}}$$

$$\frac{d}{dx}\tan^{-1} ax = \frac{a}{1+a^2x^2}$$

Prove by integrating RHS and using a trig substitution

$$\frac{d}{dx}\operatorname{cosec} ax = -(a\operatorname{cosec} ax)\cot ax$$

$$\frac{d}{dx}\cot ax = -a\operatorname{cosec}^2 ax$$

$$\int \tan x\,dx = -\ln|\cos x| + c$$

$$\int \operatorname{cosec} x\,dx = \ln\left|\tan\frac{x}{2}\right| + c$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + c$$

$$\int \cot x\,dx = \ln|\sin x| + c$$

Now if $m^2 \neq n^2$

$$\int \sin mx \sin nx\,dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}$$

$$\int \sin mx \cos nx\,dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)}$$

$$\int \cos mx \cos nx\,dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)}$$

Polynomial expansions of trigonometric functions

(WOAN pp 29)

Use **Maclaurin series**

$$\boxed{f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0)}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad (\text{all } x)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad (\text{all } x)$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \dots \quad (|x| < \pi/2)$$

$$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \dots \quad (|x| < \pi/2)$$

$$\operatorname{cosec} x = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15120} + \dots \quad (|x| < \pi)$$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \dots \quad (|x| < \pi)$$

$$\sin^{-1} x = x + \frac{x^3}{(2)(3)} + \frac{3x^5}{(2)(4)(5)} + \frac{(3)(5)x^7}{(2)(4)(6)(7)} + \dots \quad (|x| < 1)$$

$$\boxed{\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x}$$

$$\tan^{-1} x = \begin{cases} x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dots & (|x| \le 1) \\[2ex] \dfrac{\pi}{2} - \dfrac{1}{x} + \dfrac{1}{3x^2} - \dfrac{1}{5x^5} + \dots & (x > 1) \\[2ex] -\dfrac{\pi}{2} - \dfrac{1}{x} + \dfrac{1}{3x^3} - \dfrac{1}{5x^5} + \dots & (x < -1) \end{cases}$$

Note also $\boxed{\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x}$