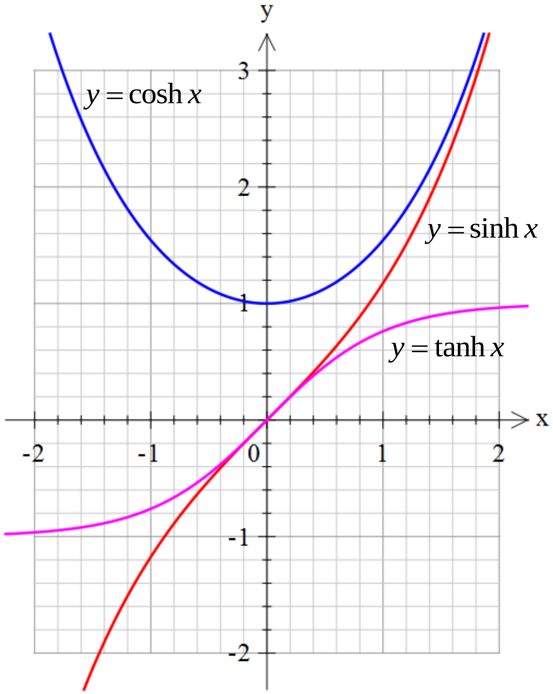
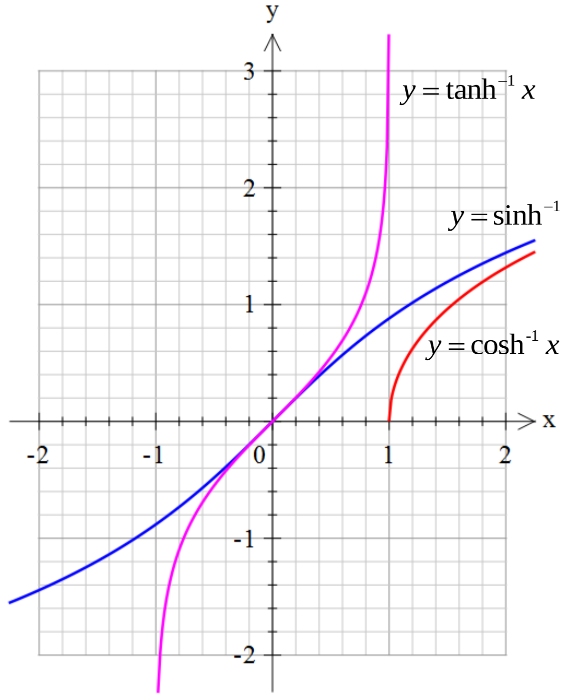
Hyperbolic functions





Inverse functions

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$
$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$$
$$\tanh^{-1} x = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$y = \sinh^{-1} x$$
$$x = \sinh y$$
$$\sinh y + \cosh y = e^y$$
$$\cosh y = \sqrt{1+\sinh^2 y} = \sqrt{1+x^2}$$
$$\therefore e^y = x + \sqrt{1+x^2}$$
$$\therefore \sinh^{-1} x = \ln\left(x + \sqrt{1+x^2}\right)$$

$\cosh^2 x - \sinh^2 x = 1$

$$y = \cosh^{-1} x$$
$$x = \cosh y$$
$$\sinh y + \cosh y = e^y$$
$$\sinh y = \sqrt{\cosh^2 y - 1} = \sqrt{x^2-1}$$
$$\therefore e^y = x + \sqrt{x^2-1}$$
$$\therefore \cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$$

$$y = \tanh^{-1} x$$
$$x = \tanh y$$
$$1 + \tanh y = 1 + \frac{e^y - e^{-y}}{e^y + e^{-y}}$$
$$1 + \tanh y = \frac{2e^y}{e^y + e^{-y}}$$
$$1 - \tanh y = 1 - \frac{e^y - e^{-y}}{e^y + e^{-y}}$$
$$1 - \tanh y = \frac{2e^{-y}}{e^y + e^{-y}}$$
$$\frac{1+x}{1-x} = \frac{2e^y}{2e^{-y}} = e^{2y}$$
$$\therefore \tanh^{-1} x = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Basic definition

$$\cosh x = \tfrac{1}{2}\left(e^x + e^{-x}\right)$$
$$\sinh x = \tfrac{1}{2}\left(e^x - e^{-x}\right)$$
$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\text{sech}\, x = \frac{1}{\cosh x}$$
$$\text{cosech}\, x = \frac{1}{\sinh x}$$
$$\coth x = \frac{1}{\tanh x}$$

Addition formulae

$$\sinh(A \pm B) = \sinh A \cosh B \pm \cosh A \sinh B$$
$$\cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B$$
$$\tanh(A \pm B) = \frac{\tanh A \pm \tanh B}{1 \pm \tanh A \tanh B}$$

$$\sinh 2x = 2\sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \tfrac{1}{2}\left(\cosh 2x + 1\right)$$
$$\sinh^2 x = \tfrac{1}{2}\left(\cosh 2x - 1\right)$$
$$\cosh^3 x = \tfrac{1}{4}\left(3\cosh x + \cosh 3x\right)$$
$$\sinh^3 x = \tfrac{1}{4}\left(-3\sinh x + \sinh 3x\right)$$

'Pythagorean' identities

$$\cosh^2 x - \sinh^2 x = 1$$
$$\text{sech}^2 x + \tanh^2 x = 1$$
$$\coth^2 x - \text{cosech}^2 x = 1$$

Take care with the sign changes!

Notice hyperbolic identities have a very similar form to ones involving sine, cosine and tangent functions. This is not surprising given the complex definitions of these functions:

$$\sin x = \tfrac{1}{2i}\left(e^{ix} - e^{-ix}\right) = -i\sinh(ix)$$
$$\cos x = \tfrac{1}{2}\left(e^{ix} + e^{-ix}\right) = \cosh(ix)$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{-i\sinh ix}{\cosh ix} = -i\tanh ix$$

Note: $\frac{1}{i} = -i$

Useful inverse relationships

$$\text{sech}^{-1} x = \cosh^{-1} \tfrac{1}{x}$$
$$\text{cosech}^{-1} x = \sinh^{-1} \tfrac{1}{x}$$
$$\coth^{-1} x = \tanh^{-1} \tfrac{1}{x}$$