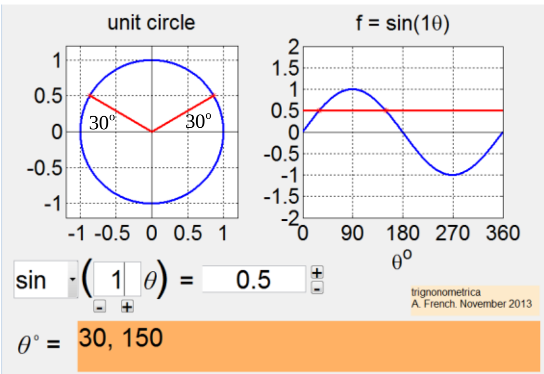
Solving Trigonometric equations

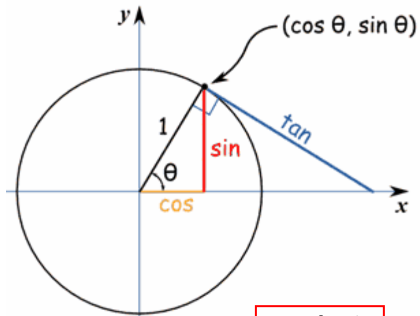
Case #1 $\sin n\theta = k$ $\cos n\theta = k$ $\tan n\theta = k$

Sine, Cosine and Tangent are all fundamentally defined by association with the **unit circle.**

If θ is the *anticlockwise angle* from the x axis, $\cos\theta$ is the ***x* coordinate** of a point on the unit circle, $\sin\theta$ is the ***y* coordinate** and $\tan\theta$ is the **length of the tangent** from this point, and also the **gradient of a diameter** of the circle through a point on the circle.

When plotted against angle θ, Sine and Cosine form *wave shaped* curves* which repeat every 360° or 2π radians. The maximum and minimum values of Sine and Cosine are, respectively, 1,-1 since these are also the maximum and minimum x and y coordinates of the unit circle.

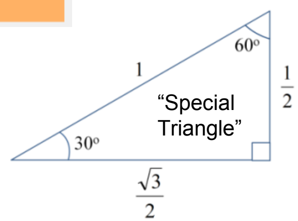




Using the unit circle definition, combined with the 'special triangle' results, it is clear that if $\sin\theta = \frac{1}{2}$

$\theta = 30^\circ + 360^\circ n \quad ; n \in \mathbb{Z}$

$\theta = 150^\circ + 360^\circ m \quad ; m \in \mathbb{Z}$



$\sin 30^\circ = \frac{1}{2}$

$\cos 30^\circ = \frac{\sqrt{3}}{2}$

$\sin 60^\circ = \frac{\sqrt{3}}{2}$

$\cos 60^\circ = \frac{1}{2}$

$\tan 30^\circ = \frac{1}{\sqrt{3}}$

$\tan 60^\circ = \sqrt{3}$

i.e. the pair of solutions indicated above repeat every 360°.

This idea can be extended to more general equations of the form $\sin n\theta = k$

$\sin 2\theta = \frac{1}{2}$

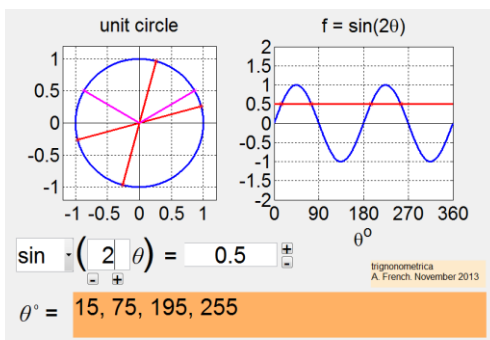
$2\theta = 30^\circ + 360^\circ n \quad ; n \in \mathbb{Z}$

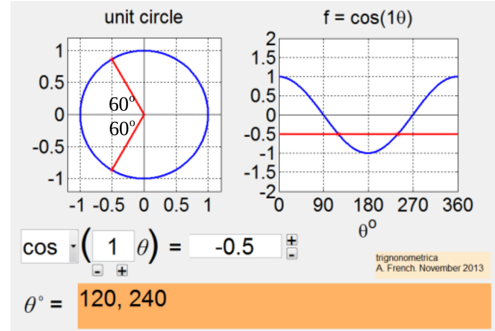
$\therefore \theta = 15^\circ + 180^\circ n \quad ; n \in \mathbb{Z}$

$2\theta = 150^\circ + 360^\circ m \quad ; m \in \mathbb{Z}$

$\therefore \theta = 75^\circ + 180^\circ m \quad ; m \in \mathbb{Z}$

In this case the anticlockwise angle is 2θ

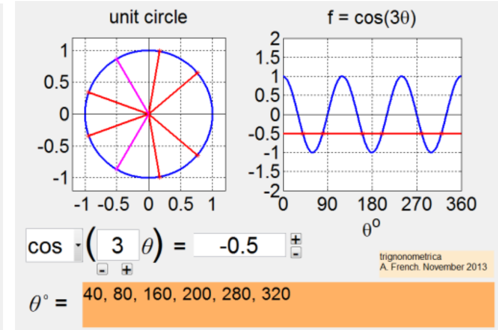




$\cos\theta = -\frac{1}{2}$

$\theta = 120^\circ + 360^\circ n \quad ; n \in \mathbb{Z}$

$\theta = 240^\circ + 360^\circ m \quad ; m \in \mathbb{Z}$



$\cos 3\theta = -\frac{1}{2}$

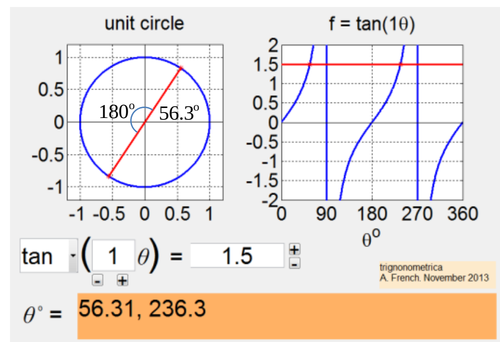
$3\theta = 120^\circ + 360^\circ n \quad ; n \in \mathbb{Z}$

$\therefore \theta = 40^\circ + 120^\circ n \quad ; n \in \mathbb{Z}$

$3\theta = 240^\circ + 360^\circ m \quad ; m \in \mathbb{Z}$

$\therefore \theta = 80^\circ + 120^\circ m \quad ; m \in \mathbb{Z}$

In this case the 'wave representation' perhaps more clearly illustrates why there are now *six* solutions in the range 0 to 360°

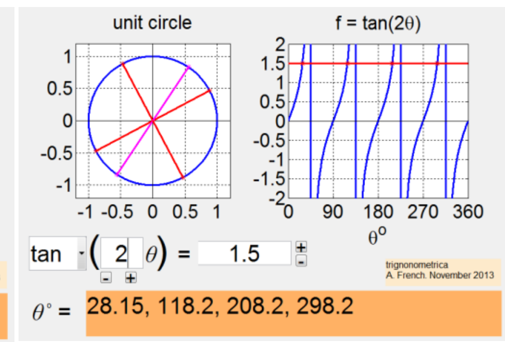


$\tan\theta = \frac{3}{2}$

$\tan^{-1}\frac{3}{2} \approx 56.31^\circ$

$\therefore \theta = 56.31^\circ + 180^\circ n \quad ; n \in \mathbb{Z}$

Since Tangent relates to a unit circle diameter, this explains why it always repeats every 180°



$\tan 2\theta = \frac{3}{2}$

$\tan^{-1}\frac{3}{2} \approx 56.31^\circ$

$\therefore 2\theta = 56.31^\circ + 180^\circ n \quad ; n \in \mathbb{Z}$

$\therefore \theta = 28.15^\circ + 90^\circ n \quad ; n \in \mathbb{Z}$

This means 'any *integer*.'
Specifically: "*n* is a *member* of the *set* of integers"

*Which explains why sine and cosine functions are used extensively in Physics to describe wave-like phenomena

Case #2 $a\sin k\theta \pm b\cos k\theta = c$

$$y = a\sin A \pm b\cos A \;\;;\;\; a, A, b > 0$$

$$R\sin\left(A \pm B\right) = R\sin A\cos B \pm R\cos A\sin B$$

Addition formula

$$a = R\cos B$$

$$b = R\sin B$$

$$\therefore \tan B = \frac{b}{a}$$

$$\therefore R = \sqrt{a^2+b^2}$$

$$\therefore a\sin A \pm b\cos B = \sqrt{a^2+b^2}\,\sin\left(A \pm \tan^{-1}\frac{b}{a}\right)$$

$$a\sin k\theta \pm b\cos k\theta = c$$

$$\therefore \sqrt{a^2+b^2}\,\sin\left(k\theta \pm \tan^{-1}\frac{b}{a}\right) = c$$

$$\therefore \sin\left(k\theta \pm \tan^{-1}\frac{b}{a}\right) = \frac{c}{\sqrt{a^2+b^2}}$$

$$\therefore k\theta \pm \tan^{-1}\frac{b}{a} = \sin^{-1}\left(\frac{c}{\sqrt{a^2+b^2}}\right) + 2\pi n \;\;;\;\; n \in \mathbb{Z} \;\;;\;\; \frac{|c|}{\sqrt{a^2+b^2}} \le 1$$

$$\therefore \theta = \frac{\sin^{-1}\left(\frac{c}{\sqrt{a^2+b^2}}\right) + 2\pi n \mp \tan^{-1}\frac{b}{a}}{k} \;\;;\;\; n \in \mathbb{Z}$$

$$k\theta \pm \tan^{-1}\frac{b}{a} = \pi - \sin^{-1}\left(\frac{c}{\sqrt{a^2+b^2}}\right) + 2\pi m \;\;;\;\; m \in \mathbb{Z}$$

$$\therefore \theta = \frac{\pi\left(2m+1\right) - \sin^{-1}\left(\frac{c}{\sqrt{a^2+b^2}}\right) \mp \tan^{-1}\frac{b}{a}}{k} \;\;;\;\; m \in \mathbb{Z}$$

No solutions if $\frac{|c|}{\sqrt{a^2+b^2}} > 1$

since this would mean values of sine beyond the unit circle

Example:

$$3\sin 2\theta - \sqrt{3}\cos 2\theta = \sqrt{3}$$

$$\therefore \sqrt{3^2+3}\,\sin\left(2\theta - \tan^{-1}\tfrac{1}{\sqrt{3}}\right) = \sqrt{3}$$

$$\therefore 2\sqrt{3}\sin\left(2\theta - \tfrac{\pi}{6}\right) = \sqrt{3}$$

$$\therefore \sin\left(2\theta - \tfrac{\pi}{6}\right) = \tfrac{1}{2}$$

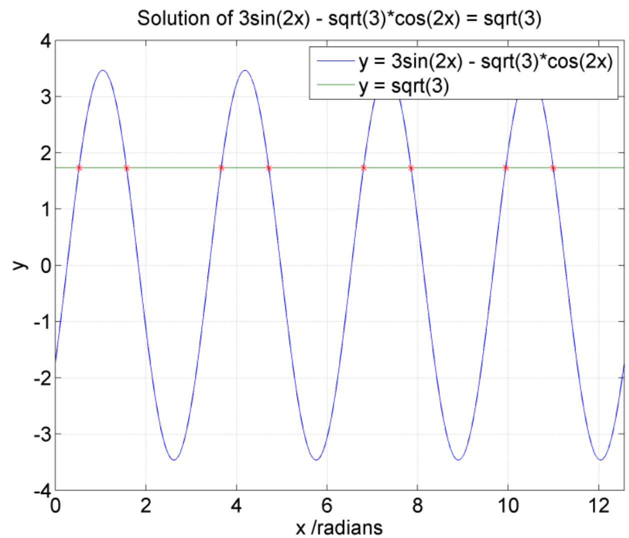
$$\therefore 2\theta - \tfrac{\pi}{6} = \tfrac{\pi}{6} + 2\pi n \;\;;\; n \in \mathbb{Z}$$

$$\therefore 2\theta = \tfrac{\pi}{3} + 2\pi n \;\;;\; n \in \mathbb{Z}$$

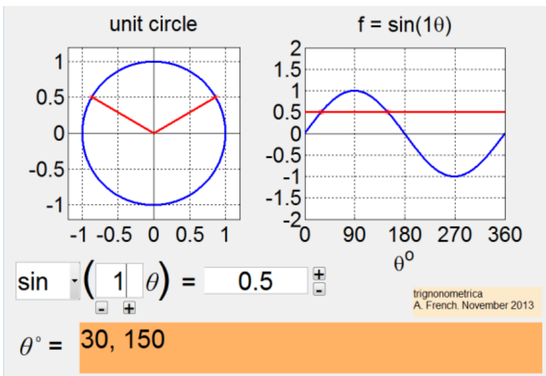
$$\therefore \theta = \left(\tfrac{1}{6} + n\right)\pi \;\;;\; n \in \mathbb{Z}$$

$$\therefore 2\theta - \tfrac{\pi}{6} = \tfrac{5\pi}{6} + 2\pi m \;\;;\; m \in \mathbb{Z}$$

$$\therefore \theta = \left(\tfrac{1}{2} + m\right)\pi \;\;;\; m \in \mathbb{Z}$$



Solutions are indicated by the red stars.



$$\theta \rightarrow 2\theta - \tfrac{\pi}{6}$$

$$30^\circ = \tfrac{\pi}{6} \text{ radians}$$

Case #3 $\sin f(\theta) = \sin g(\theta)$

$$\sin f - \sin g = 2\cos\frac{f+g}{2}\sin\frac{f-g}{2} = 0$$

i.e. 'factorizing' using a **trigonometric identity** enables the full set of solutions to be clearly identified from when each term of the 'trigonometric product' is zero.

$$\therefore \cos\frac{f+g}{2} = 0 \Rightarrow \frac{f+g}{2} = (2k+1)\tfrac{\pi}{2} \quad ; k \in \mathbb{Z}$$

$$\therefore f + g = (2k+1)\pi \quad ; k \in \mathbb{Z}$$

$$\therefore \sin\frac{f-g}{2} = 0 \Rightarrow \frac{f-g}{2} = n\pi \quad ; n \in \mathbb{Z}$$

$$\therefore f - g = 2n\pi \quad ; n \in \mathbb{Z}$$

Example:

$$\sin 3\theta = \sin(2\theta + 1)$$

$$\therefore \sin 3\theta - \sin(2\theta+1) = 0$$

$$\therefore 2\cos\frac{3\theta + 2\theta + 1}{2}\sin\frac{3\theta - 2\theta - 1}{2} = 0$$

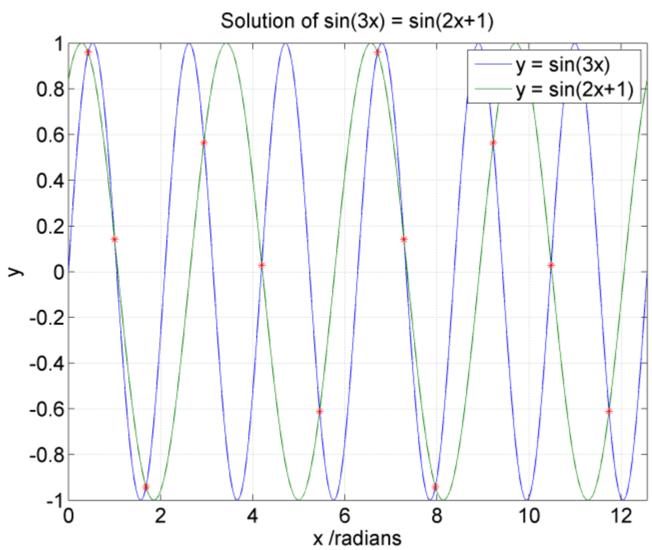
$$\therefore 2\cos\frac{5\theta+1}{2}\sin\frac{\theta-1}{2} = 0$$

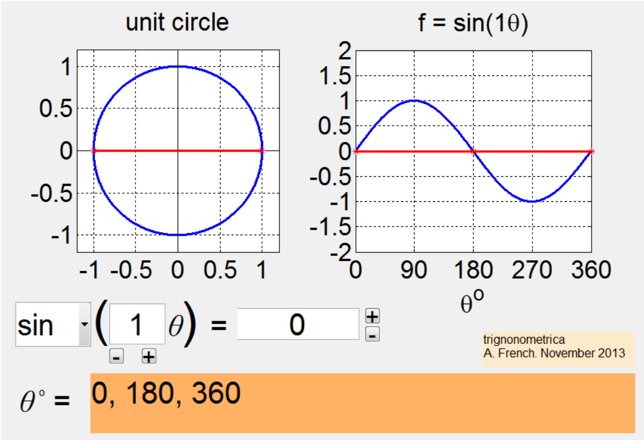
$$\therefore \cos\frac{5\theta+1}{2} = 0 \Rightarrow \frac{5\theta+1}{2} = (2k+1)\tfrac{\pi}{2} \quad ; k \in \mathbb{Z}$$

$$\therefore \theta = \frac{(2k+1)\pi - 1}{5} \quad ; k \in \mathbb{Z}$$

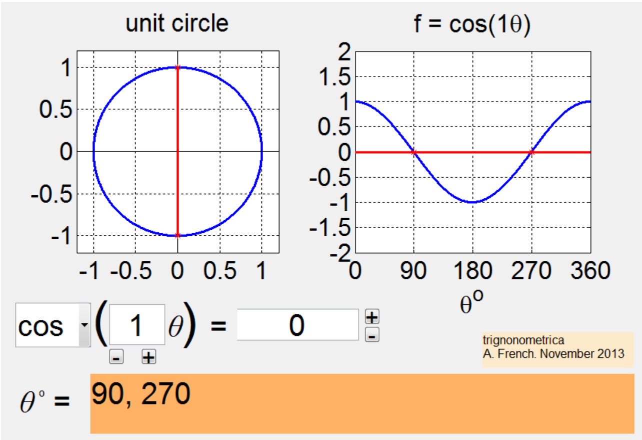
$$\therefore \sin\frac{\theta-1}{2} = 0 \Rightarrow \frac{\theta-1}{2} = n\pi \quad ; n \in \mathbb{Z}$$

$$\therefore \theta = 2n\pi + 1 \quad ; n \in \mathbb{Z}$$





$$\sin\theta = 0$$
$$\therefore \theta = n\pi \quad ; n \in \mathbb{Z}$$



$$\cos\theta = 0$$
$$\therefore \theta = (2k+1)\tfrac{\pi}{2} \quad ; k \in \mathbb{Z}$$

Case #4 $\sin f(\theta) = -\sin g(\theta)$

$$\sin f + \sin g = 2\sin\frac{f+g}{2}\cos\frac{f-g}{2} = 0$$

$$\therefore \cos\frac{f+g}{2} = 0 \Rightarrow \frac{f+g}{2} = (2k+1)\tfrac{\pi}{2} \quad ; k \in \mathbb{Z}$$

$$\therefore f + g = (2k+1)\pi \quad ; k \in \mathbb{Z}$$

$$\therefore \sin\frac{f-g}{2} = 0 \Rightarrow \frac{f-g}{2} = n\pi \quad ; n \in \mathbb{Z}$$

$$\therefore f - g = 2n\pi \quad ; n \in \mathbb{Z}$$

Example:

$$\sin 5\theta = -\sin\left(\theta - \tfrac{1}{4}\pi\right)$$

$$\therefore \sin 5\theta + \sin\left(\theta - \tfrac{1}{4}\pi\right) = 0$$

$$\therefore 2\sin\frac{5\theta + \theta - \frac{1}{4}\pi}{2}\cos\frac{5\theta - \theta + \frac{1}{4}\pi}{2} = 0$$

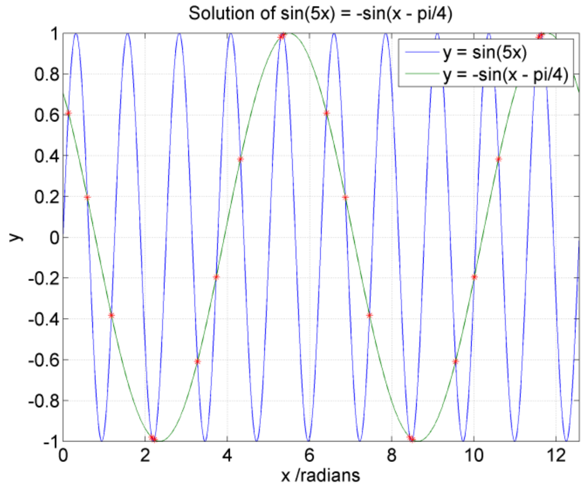
$$\therefore 2\sin\frac{6\theta - \frac{1}{4}\pi}{2}\cos\frac{4\theta + \frac{1}{4}\pi}{2} = 0$$

$$\sin\frac{6\theta - \frac{1}{4}\pi}{2} = 0 \quad \therefore \frac{6\theta - \frac{1}{4}\pi}{2} = n\pi; \; n \in \mathbb{Z}$$

$$\therefore \theta = \frac{2n\pi + \frac{1}{4}\pi}{6} \quad ; n \in \mathbb{Z}$$

$$\cos\frac{4\theta + \frac{1}{4}\pi}{2} = 0 \quad \therefore \frac{4\theta + \frac{1}{4}\pi}{2} = (2k+1)\tfrac{\pi}{2} \quad ; k \in \mathbb{Z}$$

$$\therefore \theta = \frac{(2k+1)\pi - \frac{1}{4}\pi}{4} \quad ; n \in \mathbb{Z}$$



Case #5 $\cos f(\theta) = \cos g(\theta)$

$$\cos f - \cos g = -2\sin\frac{f+g}{2}\sin\frac{f-g}{2} = 0$$

$$\therefore \sin\frac{f \pm g}{2} = 0 \Rightarrow \frac{f-g}{2} = n\pi \quad ; n \in \mathbb{Z}$$

$$\therefore f + g = 2n\pi \quad ; n \in \mathbb{Z}$$

$$\therefore f - g = 2m\pi \quad ; m \in \mathbb{Z}$$

Example:

$$\cos 3\theta = \cos 2\theta$$

$$\therefore \cos 3\theta - \cos 2\theta = 0$$

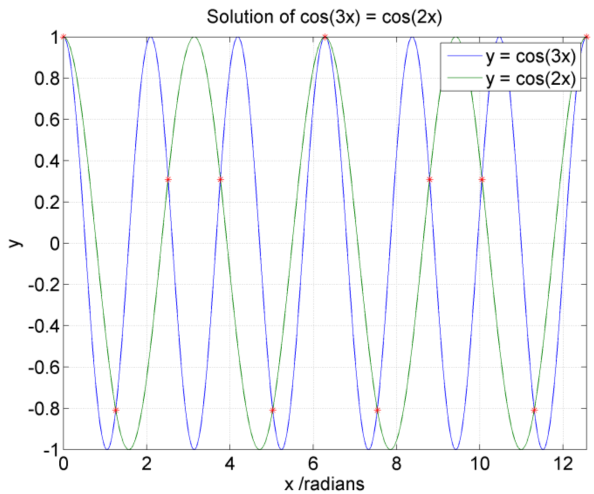
$$\therefore -2\sin\tfrac{5}{2}\theta\sin\tfrac{1}{2}\theta = 0$$

$$\sin\tfrac{5}{2}\theta = 0 \Rightarrow \tfrac{5}{2}\theta = n\pi \quad ; \quad n \in \mathbb{Z}$$

$$\theta = \tfrac{2}{5}n\pi \quad ; \quad n \in \mathbb{Z}$$

$$\sin\tfrac{1}{2}\theta = 0 \Rightarrow \tfrac{1}{2}\theta = m\pi \quad ; \quad n \in \mathbb{Z}$$

$$\therefore \theta = 2m\pi \quad ; \quad m \in \mathbb{Z}$$



Case #6 $\cos f(\theta) = -\cos g(\theta)$

$$\cos f + \cos g = 2\cos\frac{f+g}{2}\cos\frac{f-g}{2} = 0$$

$$\therefore \cos\frac{f+g}{2} = 0 \Rightarrow \frac{f+g}{2} = (2k+1)\tfrac{\pi}{2} \quad ; k \in \mathbb{Z}$$

$$\therefore f + g = (2k+1)\pi \quad ; k \in \mathbb{Z}$$

$$\therefore \cos\frac{f-g}{2} = 0 \Rightarrow \frac{f-g}{2} = (2n+1)\tfrac{\pi}{2} \quad ; n \in \mathbb{Z}$$

$$\therefore f - g = (2n+1)\pi \quad ; n \in \mathbb{Z}$$

Example:

$$\cos(2\theta + 1) = -\cos(\theta - 1)$$

$$\therefore \cos(2\theta + 1) + \cos(\theta - 1) = 0$$

$$\therefore 2\cos\tfrac{3}{2}\theta\cos\tfrac{1}{2}(\theta + 2) = 0$$

$$\cos\tfrac{3}{2}\theta = 0$$

$$\therefore \tfrac{3}{2}\theta = (2k+1)\tfrac{\pi}{2} \quad ; \quad k \in \mathbb{Z}$$

$$\therefore \theta = (2k+1)\tfrac{\pi}{3} \quad ; \quad k \in \mathbb{Z}$$

$$\cos\tfrac{1}{2}(\theta + 2) = 0$$

$$\therefore \tfrac{1}{2}(\theta + 2) = (2n+1)\tfrac{\pi}{2} \quad ; \quad n \in \mathbb{Z}$$

$$\therefore \theta = (2n+1)\pi - 2 \quad ; \quad n \in \mathbb{Z}$$