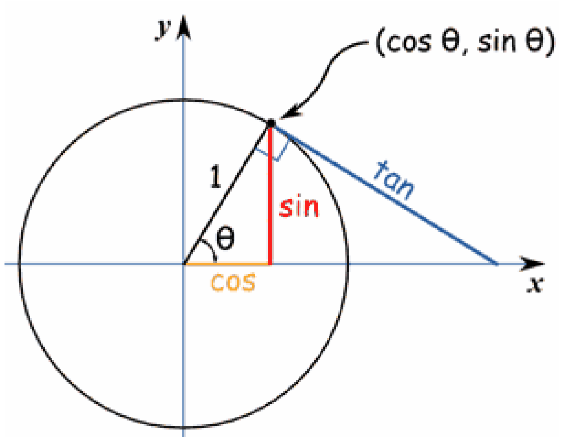
Trigonometry and the unit circle

The *unit circle* can be used to extend the meaning of basic trigonometric functions *beyond* the domain of angles $0 < \theta < 90^o$.

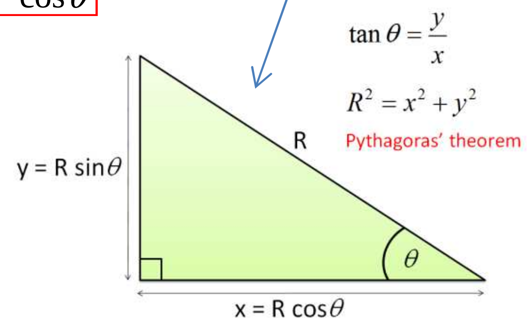


$\cos\theta$ **is the x coordinate of the unit circle**
$\sin\theta$ **is the y coordinate of the unit circle**

θ **is measured anti-clockwise from the x axis**

$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$

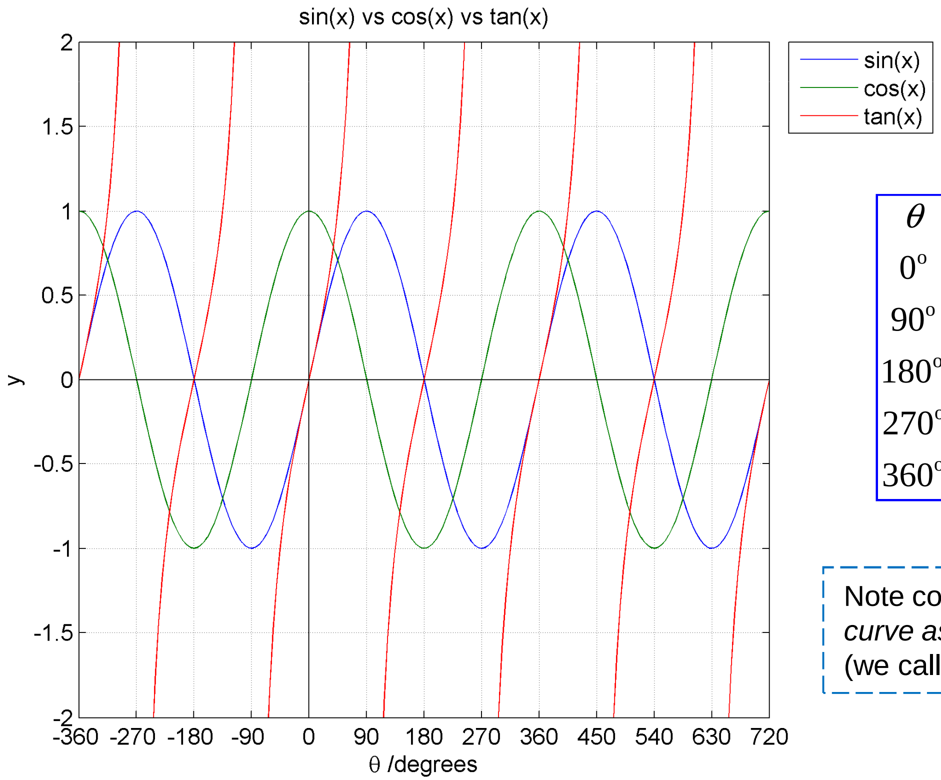
Basic trigonometry defined for right angled triangles



We can use this definition to extend the definition of sine and cosine to *any* angle θ.

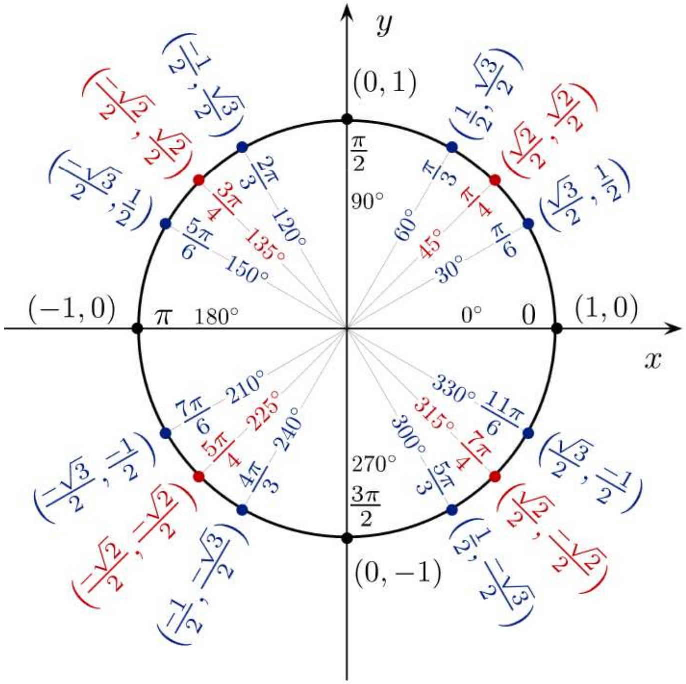
When plotted, sine and cosine are smoothly varying functions between -1 and 1, which repeat every 360º.

$\tan\theta$ becomes *infinite* at 90º and repeats every 180º.



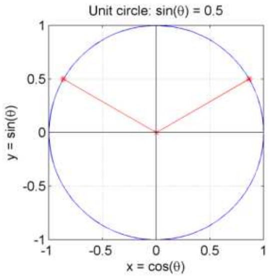
θ	$x = \cos\theta$	$y = \sin\theta$
0^o	1	0
90^o	0	1
180^o	-1	0
270^o	0	-1
360^o	1	0

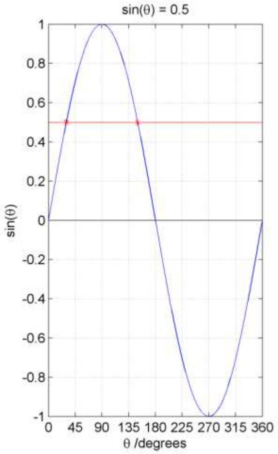
Note cosine is the *same curve as sine*, but 'lagging behind' (we call this *phase shifted*) by 90º.



(*x*,*y*) coordinates of the unit circle vs anti-clockwise angle θ measured from the *x* axis. The middle number is the angle measured in *radians*. The conversion is π radians = 180º.

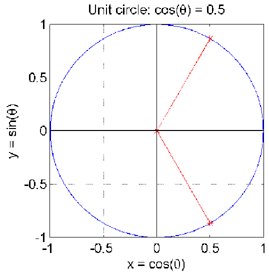
Trigonometric functions are *periodic*, i.e. repeat after a fixed interval. This means the solutions to trigonometric equations like sin(x) = 0.5 are *multi-valued.*

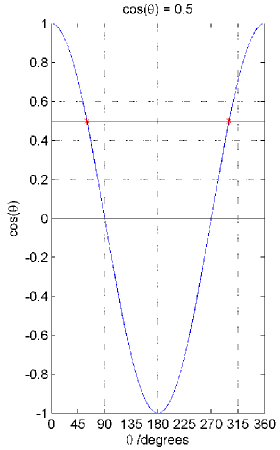




$\sin\theta = \frac{1}{2}$

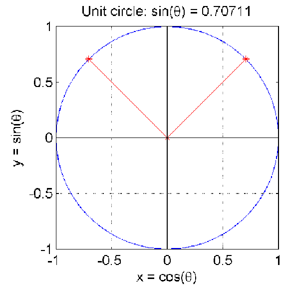
$\theta = 30^o, 150^o, \ldots$

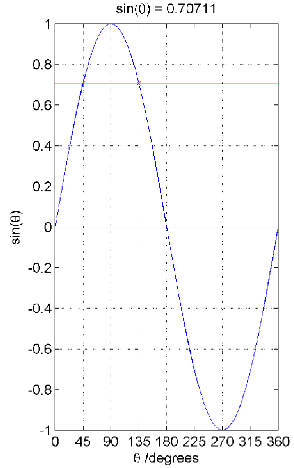




$\cos\theta = \frac{1}{2}$

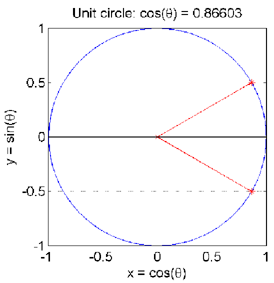
$\theta = 60^o, 300^o, \ldots$

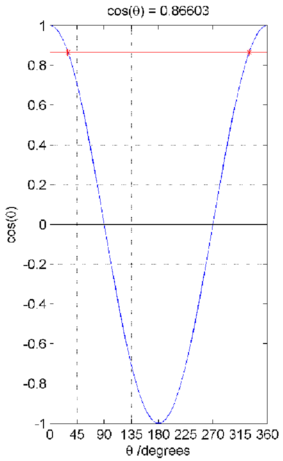




$\sin\theta = \frac{1}{\sqrt{2}}$

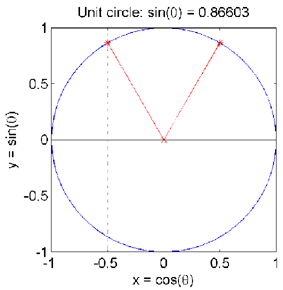
$\theta = 45^o, 135^o, \ldots$

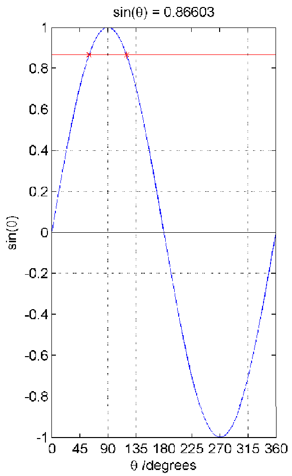




$\cos\theta = \frac{\sqrt{3}}{2}$

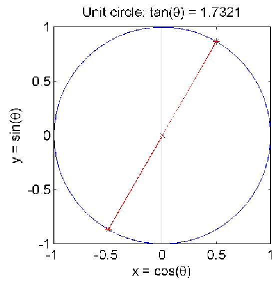
$\theta = 30^o, 330^o, \ldots$

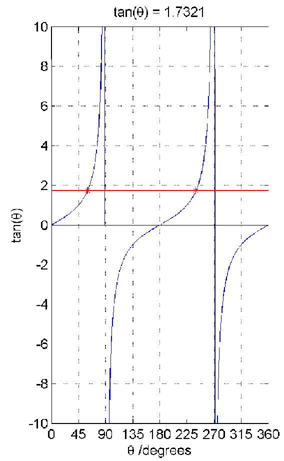




$\sin\theta = \frac{\sqrt{3}}{2}$

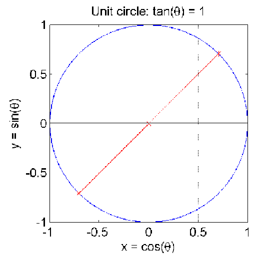
$\theta = 60^o, 120^o, \ldots$

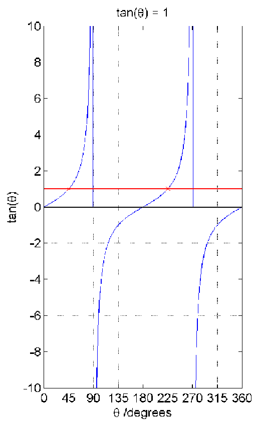




$\tan\theta = \sqrt{3}$

$\theta = 60^o, 240^o, \ldots$

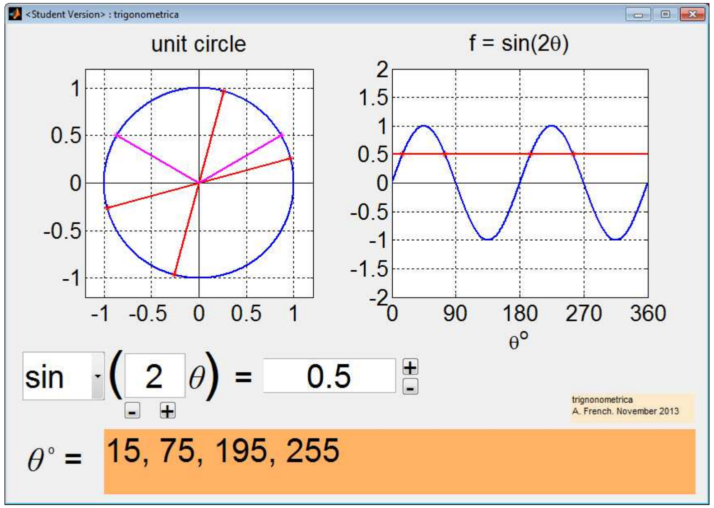


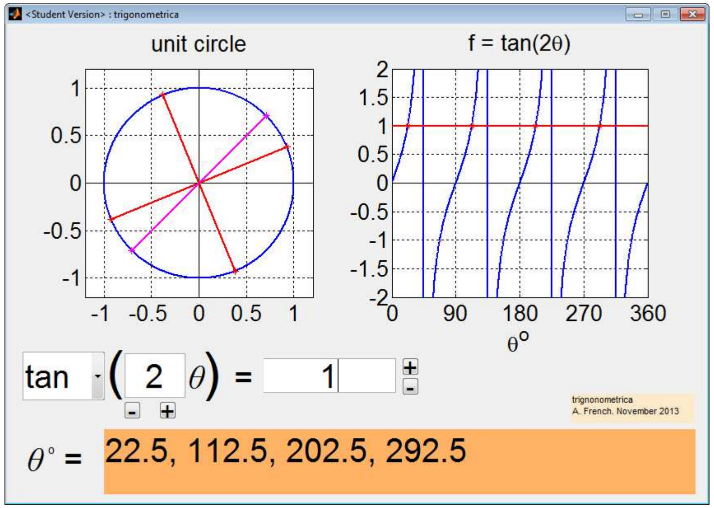


$\tan\theta = 1$

$\theta = 45^o, 225^o, \ldots$

Note: all the graphs above give solutions in the range 0º 360º



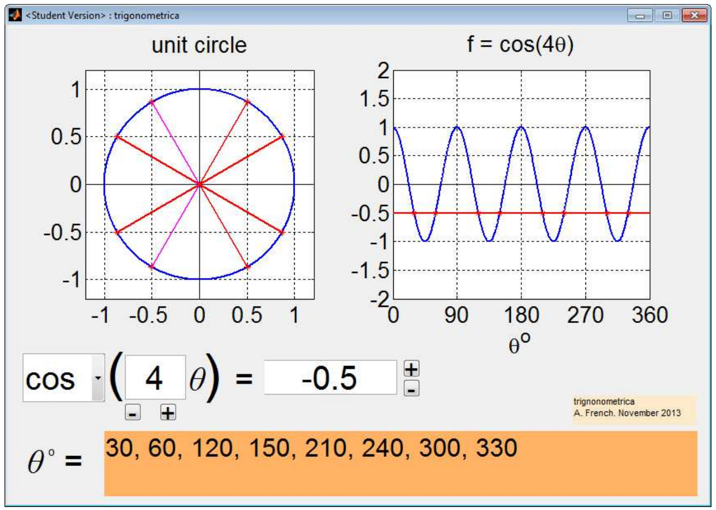


$$\sin 2\theta = \tfrac{1}{2}$$
$$\therefore 2\theta = 30^{\circ} + 360N$$
$$\therefore 2\theta = 150^{\circ} + 360M$$
$$\Rightarrow \boxed{\theta = 15^{\circ} + 180N}$$
$$\Rightarrow \boxed{\theta = 75^{\circ} + 180M}$$

For this type of problem one must consider integer multiples of a full circumnavigation of the unit circle in order to include all the repeated solutions.

$$\tan 2\theta = 1$$
$$\therefore 2\theta = 45^{\circ} + 180N$$
$$\Rightarrow \boxed{\theta = 22.5^{\circ} + 90N}$$

Note tan($n\theta$) is the *gradient* of a diameter of the unit circle. The clockwise angle $n\theta$ therefore repeats every 180°.



$$\cos 4\theta = -\tfrac{1}{2}$$
$$\therefore 4\theta = 120^{\circ} + 360N$$
$$\therefore 4\theta = 240^{\circ} + 360M$$
$$\Rightarrow \boxed{\theta = 30^{\circ} + 90N \quad \text{i.e. } 30^{\circ}, 120^{\circ}, 210^{\circ}, 300^{\circ} \ldots}$$
$$\Rightarrow \boxed{\theta = 60^{\circ} + 90M \quad \text{i.e. } 60^{\circ}, 150^{\circ}, 240^{\circ}, 330^{\circ} \ldots}$$