Trigonometric (and hyperbolic) identities via complex numbers

We can use *De-Moivre's Theorem* to express powers of trigonometric functions in terms of unity power terms of higher frequency. The latter can be readily integrated.

Firstly, let us write sine and cosine in terms of a complex number z of unit magnitude

$$z = e^{i\theta}, \ \frac{1}{z} = e^{i\theta}$$
$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\therefore 2\cos\theta = e^{i\theta} + e^{-i\theta}$$

$$\cos\theta = \tfrac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$
$$2\cos\theta = z + \frac{1}{z}$$

$$\therefore 2i\sin\theta = e^{i\theta} - e^{-i\theta}$$

$$\sin\theta = \tfrac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$
$$2i\sin\theta = z - \frac{1}{z}$$

Hence:

$$2\cos n\theta = e^{in\theta} + e^{-in\theta}$$

$$2\cos n\theta = z^n + \frac{1}{z^n}$$

$$2i\sin n\theta = e^{in\theta} - e^{-in\theta}$$

$$2i\sin n\theta = z^n - \frac{1}{z^n}$$

We can use these results to determine **identities** involving powers of sine and cosine e.g.

$$\sin^3\theta = \tfrac{1}{4}\left(3\sin\theta - \sin 3\theta\right)$$

Example #1:

$$(2i\sin\theta)^3 = \left(z - \frac{1}{z}\right)^3$$
$$2i\sin\theta = z - \frac{1}{z}$$
$$2i\sin 2\theta = z^2 - \frac{1}{z^2}$$
$$2i\sin 3\theta = z^3 - \frac{1}{z^3}$$
$$\left(z - \frac{1}{z}\right)^3 = z^3 + 3z^2\left(-\frac{1}{z}\right) + 3z\left(-\frac{1}{z}\right)^2 + \left(-\frac{1}{z}\right)^3$$

Binomial expansion

$$\left(z - \frac{1}{z}\right)^3 = z^3 - 3z + \frac{3}{z} - \frac{1}{z^3}$$
$$\left(z - \frac{1}{z}\right)^3 = z^3 - \frac{1}{z^3} - 3\left(z - \frac{1}{z}\right)$$
$$(2i\sin\theta)^3 = 2i\sin 3\theta - 3(2i\sin\theta)$$
$$-8i\sin^3\theta = 2i\sin 3\theta - 6i\sin\theta$$
$$\sin^3\theta = \tfrac{1}{4}\left(3\sin\theta - \sin 3\theta\right)$$

$$\therefore \int \sin^3\theta \, d\theta = \int \tfrac{1}{4}\left(3\sin\theta - \sin 3\theta\right) d\theta$$
$$\therefore \int \sin^3\theta \, d\theta = \tfrac{1}{4}\left(-3\cos\theta + \tfrac{1}{3}\cos 3\theta\right) + c$$
$$\therefore \int \sin^3\theta \, d\theta = \tfrac{1}{12}\cos 3\theta - \tfrac{3}{4}\cos\theta + c$$

Example #2:

$$(2\cos\theta)^5 = \left(z + \frac{1}{z}\right)^5$$
$$2\cos 2\theta = z^2 + \frac{1}{z^2}$$
$$2\cos 5\theta = z^5 + \frac{1}{z^5}$$

....

$$\left(z + \frac{1}{z}\right)^5 = z^5 + 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z}\right)^2 + 10z^2\left(\frac{1}{z}\right)^3 + 5z\left(\frac{1}{z}\right)^4 + \left(\frac{1}{z}\right)^5$$

Binomial expansion

$$\left(z + \frac{1}{z}\right)^5 = z^5 + 5z^3 + 10z + \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5}$$
$$\left(z + \frac{1}{z}\right)^5 = z^5 + \frac{1}{z^5} + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right)$$
$$(2\cos\theta)^5 = 2\cos 5\theta + 5(2\cos 3\theta) + 10(2\cos\theta)$$
$$32\cos^5\theta = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$$

$$\therefore \cos^5\theta = \tfrac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right)$$
$$\therefore \int \cos^5\theta \, d\theta = \int \tfrac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right) d\theta$$
$$\therefore \int \cos^5\theta \, d\theta = \tfrac{1}{16}\left(\tfrac{1}{5}\sin 5\theta + \tfrac{5}{3}\sin 3\theta + 10\sin\theta\right) + c$$
$$\therefore \int \cos^5\theta \, d\theta = \tfrac{1}{80}\sin 5\theta + \tfrac{5}{48}\sin 3\theta + \tfrac{5}{8}\sin\theta + c$$

The sum of a **Geometric Progression** can also be combined with these results

$$1 + z + z^2 + z^3 + z^4 + ... + z^{n-1} = \frac{1 - z^n}{1 - z}$$

$$z = e^{i\theta} \quad \therefore z^n = e^{in\theta}$$
$$z = \cos\theta + i\sin\theta \quad \therefore z^n = \cos n\theta + i\sin n\theta$$
$$1 + (\cos\theta + i\sin\theta) + (\cos\theta + i\sin\theta)^2 + ...(\cos\theta + i\sin\theta)^{n-1} = \frac{1 - (\cos\theta + i\sin\theta)^n}{1 - \cos\theta - i\sin\theta}$$
$$1 + \cos\theta + \cos 2\theta + ...\cos(n-1)\theta$$
$$+i\left(\sin\theta + \sin 2\theta + ...\sin(n-1)\theta\right) = \frac{1 - \cos n\theta - i\sin n\theta}{1 - \cos\theta - i\sin\theta} \times \frac{1 - \cos\theta + i\sin\theta}{1 - \cos\theta + i\sin\theta}$$

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$$1+\cos\theta+\cos 2\theta+....\cos(n-1)\theta+i(\sin\theta+\sin 2\theta+....\sin(n-1)\theta)=\frac{1-\cos n\theta-i\sin n\theta}{1-\cos\theta-i\sin\theta}\times\frac{1-\cos\theta+i\sin\theta}{1-\cos\theta+i\sin\theta}$$

$$\frac{1-\cos n\theta-i\sin n\theta}{1-\cos\theta-i\sin\theta}\times\frac{1-\cos\theta+i\sin\theta}{1-\cos\theta+i\sin\theta}$$
$$=\frac{(1-\cos n\theta)(1-\cos\theta)+i\sin\theta(1-\cos n\theta)-i\sin n\theta(1-\cos\theta)+\sin\theta\sin n\theta}{(1-\cos\theta)^2+\sin^2\theta}$$
$$=\frac{(1-\cos n\theta)(1-\cos\theta)+\sin\theta\sin n\theta+i\sin\theta(1-\cos n\theta)-i\sin n\theta(1-\cos\theta)}{2-2\cos\theta}$$
$$=\tfrac{1}{2}(1-\cos n\theta)+\frac{\sin\theta\sin n\theta}{2-2\cos\theta}+i\left\{\frac{\sin\theta(1-\cos n\theta)}{2-2\cos\theta}-\tfrac{1}{2}\sin n\theta\right\}$$

Hence by equating real and imaginary parts:

$$1+\cos\theta+\cos 2\theta+....\cos(n-1)\theta=\tfrac{1}{2}(1-\cos n\theta)+\tfrac{1}{2}\frac{\sin\theta\sin n\theta}{1-1\cos\theta}$$
$$\sin\theta+\sin 2\theta+....\sin(n-1)\theta=\tfrac{1}{2}\frac{\sin\theta(1-\cos n\theta)}{1-1\cos\theta}-\tfrac{1}{2}\sin n\theta$$

Using summation notation:

$$\sum_{k=1}^{n}\cos(k-1)\theta=\tfrac{1}{2}(1-\cos n\theta)+\tfrac{1}{2}\frac{\sin\theta\sin n\theta}{1-1\cos\theta}$$
$$\sum_{k=1}^{n}\sin(k-1)\theta=\tfrac{1}{2}\frac{\sin\theta(1-\cos n\theta)}{1-1\cos\theta}-\tfrac{1}{2}\sin n\theta$$

Similar ideas can be used for **Hyperbolic functions**

$$\cosh nx=\tfrac{1}{2}(e^{nx}+e^{-nx})$$
$$\sinh nx=\tfrac{1}{2}(e^{nx}-e^{-nx})$$
$$z=e^x$$
$$\therefore 2\cosh nx=z^n+\frac{1}{z^n}$$
$$\therefore 2\sinh nx=z^n-\frac{1}{z^n}$$

Example #3

$$(2\cosh x)^4=\left(z+\frac{1}{z}\right)^4$$
$$=z^4+4z^3\frac{1}{z}+6z^2\frac{1}{z^2}+4z\frac{1}{z^3}+\frac{1}{z^4}$$
$$=z^4+4z^2+6+\frac{4}{z^2}+\frac{1}{z^4}$$
$$=z^4+\frac{1}{z^4}+4\left(z^2+\frac{1}{z^2}\right)+6$$
$$=2\cosh 4x+4(2\cosh 2x)+6$$
$$\therefore 16\cosh^4 x=2\cosh 4x+8\cosh 2x+6$$

$$\therefore \cosh^4 x=\tfrac{1}{8}(\cosh 4x+4\cosh 2x+3)$$

$$\int\cosh^4 x\,dx=\int\tfrac{1}{8}(\cosh 4x+4\cosh 2x+3)\,dx$$
$$\int\cosh^4 x\,dx=\tfrac{1}{32}\sinh 4x+\tfrac{1}{4}\sinh 2x+\tfrac{3}{8}x+c$$

Example #4

$$(2\sinh x)^3=\left(z-\frac{1}{z}\right)^3$$
$$=z^3+3z^2\left(-\frac{1}{z}\right)+3z\left(-\frac{1}{z}\right)^2+\left(-\frac{1}{z}\right)^3$$
$$=z^3-3z+\frac{3}{z}-\frac{1}{z^3}$$
$$8\sinh^3 x=z^3-\frac{1}{z^3}-3\left(z-\frac{1}{z}\right)$$
$$8\sinh^3 x=2\sinh 3x-3(2\sinh x)$$
$$\therefore \sinh^3 x=\tfrac{1}{4}\sinh 3x-\tfrac{3}{4}\sinh x$$

$$\therefore\int\sinh^3 x\,dx=\int\left(\tfrac{1}{4}\sinh 3x-\tfrac{3}{4}\sinh x\right)dx$$
$$\therefore\int\sinh^3 x\,dx=\tfrac{1}{12}\cosh 3x-\tfrac{3}{4}\cosh x+c$$