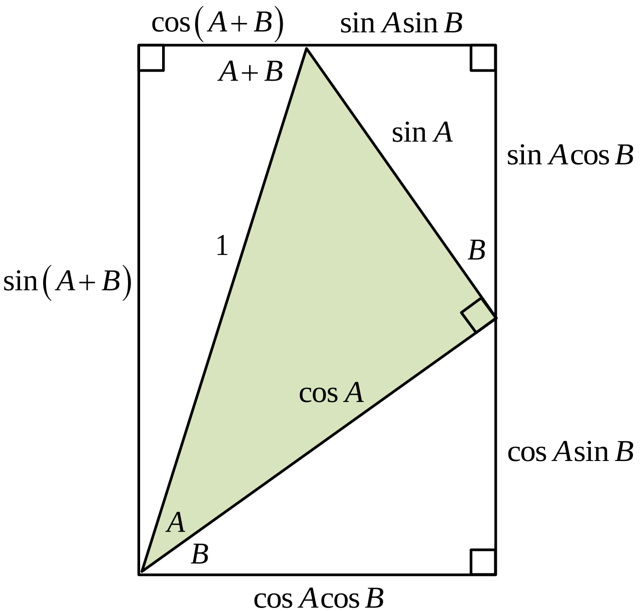
Trigonometric identities

Useful algebraic relationships can be proven between the basic trigonometric functions of sine, cosine and tangent.

Geometric proof of the addition formulae



$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

odd and even relationship

$$\cos(\pm A) = \cos A$$
$$\sin(\pm A) = \pm \sin A$$

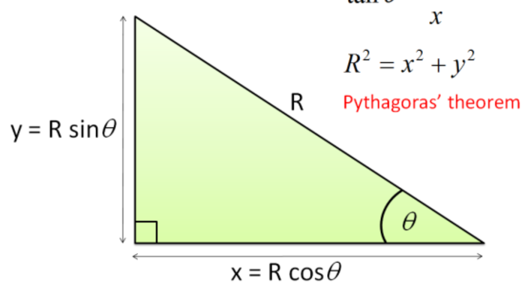
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

Addition formulae for tan

$$\tan(A \pm B) = \frac{\sin A \cos B \pm \cos A \sin B}{\cos A \cos B \mp \sin A \sin B}$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Basic trigonometry



$$\tan\theta = \frac{y}{x}$$

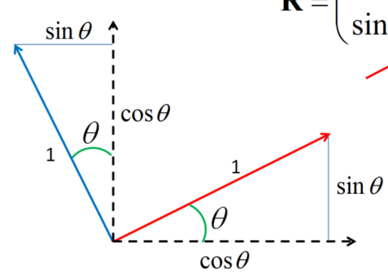
$$R^2 = x^2 + y^2$$

Pythagoras' theorem

$$\sin^2\theta + \cos^2\theta = 1$$

Rotation anticlockwise about (0,0) by angle θ

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$



$$\sin(A+A) = \sin A \cos A + \cos A \sin A$$
$$\cos(A+A) = \cos A \cos A - \sin A \sin A$$
$$\tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

Double angle formulae

$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Squares of sine and cosine

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$\cos 2A = 1 - \sin^2 A - \sin^2 A$$
$$\cos 2A = 1 - 2\sin^2 A$$
$$\sin^2 A = \tfrac{1}{2}(1 - \cos 2A)$$

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$\cos 2A = \cos^2 A - (1 - \cos^2 A)$$
$$\cos 2A = 2\cos^2 A - 1$$
$$\cos^2 A = \tfrac{1}{2}(1 + \cos 2A)$$

Use of rotation matrices to derive sine and cosine addition formulae

$$\mathbf{R}_{A \pm B} = \mathbf{R}_{\pm B} \mathbf{R}_A$$

$$\begin{pmatrix} \cos(A \pm B) & -\sin(A \pm B) \\ \sin(A \pm B) & \cos(A \pm B) \end{pmatrix}$$
$$= \begin{pmatrix} \cos B & \mp \sin B \\ \pm \sin B & \cos B \end{pmatrix} \begin{pmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{pmatrix}$$

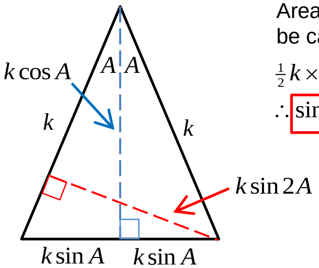
$$\therefore$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

Formulae for cubes of sine and cosine

$$\sin(2A+A)=\sin 2A\cos A+\cos 2A\sin A$$
$$\cos(2A+A)=\cos 2A\cos A-\sin 2A\sin A$$

$$\sin 2A=2\sin A\cos A$$
$$\cos 2A=\cos^2 A-\sin^2 A$$



Area of an isosceles triangle can be calculated in two ways:

$$\tfrac{1}{2}k\times k\sin 2A=2\times\tfrac{1}{2}k\sin A\times k\cos A$$
$$\therefore \boxed{\sin 2A=2\sin A\cos A}$$

$$\sin 3A=2\sin A\cos A\cos A+\left(\cos^2 A-\sin^2 A\right)\sin A$$
$$\sin 3A=2\sin A\cos^2 A+\cos^2 A\sin A-\sin^3 A$$
$$\sin 3A=3\sin A\cos^2 A-\sin^3 A$$
$$\sin 3A=3\sin A\left(1-\sin^2 A\right)-\sin^3 A$$
$$\sin 3A=3\sin A-4\sin^3 A$$
$$\boxed{\sin^3 A=\tfrac{1}{4}(3\sin A-\sin 3A)}$$

$$\cos 3A=\left(\cos^2 A-\sin^2 A\right)\cos A-2\sin A\cos A\sin A$$
$$\cos 3A=\left(\cos^2 A-\left(1-\cos^2 A\right)\right)\cos A-2\sin A\cos A\sin A$$
$$\cos 3A=\left(2\cos^2 A-1\right)\cos A-2\sin^2 A\cos A$$
$$\cos 3A=2\cos^3 A-\cos A-2\left(1-\cos^2 A\right)\cos A$$
$$\cos 3A=2\cos^3 A-\cos A-2\cos A+2\cos^3 A$$
$$\cos 3A=4\cos^3 A-3\cos A$$
$$\boxed{\cos^3 A=\tfrac{1}{4}(3\cos A+\cos 3A)}$$

Addition formulae for inverses

$$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$$
$$A\pm B=\sin^{-1}\left(\sin A\sqrt{1-\sin^2 B}\pm\sqrt{1-\sin^2 A}\sin B\right)$$
$$a=\sin A$$
$$b=\sin B$$
$$\boxed{\sin^{-1}a\pm\sin^{-1}b=\sin^{-1}\left(a\sqrt{1-b^2}\pm b\sqrt{1-a^2}\right)}$$

$$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$$
$$A\pm B=\cos^{-1}\left(\cos A\cos B\mp\sqrt{1-\cos^2 A}\sqrt{1-\cos^2 B}\right)$$
$$a=\cos A$$
$$b=\cos B$$
$$\boxed{\cos^{-1}a\pm\cos^{-1}b=\cos^{-1}\left(ab\mp\sqrt{1-a^2}\sqrt{1-b^2}\right)}$$

$$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$$
$$A\pm B=\tan^{-1}\left(\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}\right)$$
$$a=\tan A$$
$$b=\tan B$$
$$\boxed{\tan^{-1}a\pm\tan^{-1}b=\tan^{-1}\left(\frac{a\pm b}{1\mp ab}\right)}$$

Addition of mixed sine and cosine

$$a\sin A\pm b\cos A$$
$$R\sin(A\pm\alpha)=R\sin A\cos\alpha\pm R\cos A\sin\alpha$$

$$R\cos\alpha=a$$
$$R\sin\alpha=b$$

$$\tan\alpha=\frac{b}{a}$$
$$R^2\cos^2\alpha+R^2\sin^2\alpha=a^2+b^2$$
$$R=\sqrt{a^2+b^2}$$

$$\boxed{a\sin A\pm b\cos A=\sqrt{a^2+b^2}\sin\left(A\pm\tan^{-1}\tfrac{b}{a}\right)}$$

Other useful identities provable from addition formulae

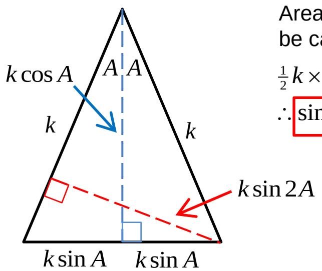
$$\cos A\cos B=\frac{1}{2}[\cos(A+B)+\cos(A-B)]$$
$$\sin A\cos B=\frac{1}{2}[\sin(A+B)+\sin(A-B)]$$
$$\sin A\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$$

$$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$
$$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

Alternative proof for sine addition formula

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$

Motivation from *double angle formula proof*:

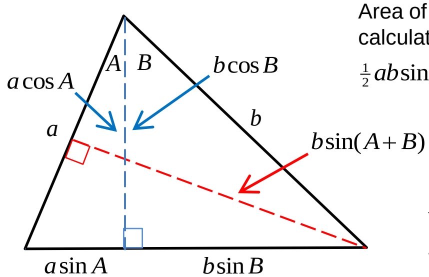


Area of an isosceles triangle can be calculated in two ways:

$\frac{1}{2}k\times k\sin 2A = 2\times\frac{1}{2}k\sin A\times k\cos A$

$\therefore \boxed{\sin 2A = 2\sin A\cos A}$

Now generalize!



Area of the larger triangle can be calculated in two ways:

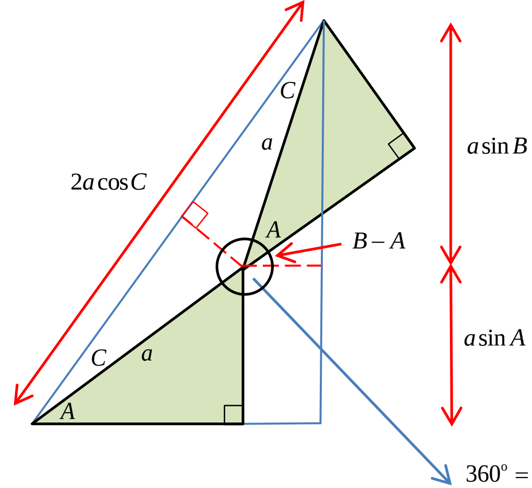
$\frac{1}{2}ab\sin(A+B)=\frac{1}{2}a\sin A\times h \quad +\frac{1}{2}b\sin B\times h$

Note vertical height: $h=b\cos B=a\cos A$

$\frac{1}{2}ab\sin(A+B)=\frac{1}{2}a\sin A\times b\cos B+\frac{1}{2}b\sin B\times a\cos A$

$\therefore \boxed{\sin(A+B)=\sin A\cos B+\cos A\sin B}$

Note: you'll need a and b to be the lengths of the slanted sides, and choose the form of the vertical height of the triangle h such that ab is a factor of each term on the RHS, which will then cancel with the ab factor on the LHS.



$a\sin A+a\sin B=2a\cos C\sin(A+C)$

$\therefore \sin A+\sin B=2\cos C\sin(A+C)$

$$\boxed{\sin A+\sin B=2\cos\left(\tfrac{B-A}{2}\right)\sin\left(\tfrac{B+A}{2}\right)}$$

$360^\circ = 90^\circ - A + 90^\circ + B + 90^\circ - C + 90^\circ - C$

$0 = B - A - 2C$

$C=\frac{1}{2}(B-A)$