Sine Rule & Cosine Rule  These are two extremely useful trigonometric results which are applicable to all triangles, not just right angled ones.

To prove the Sine Rule, consider three identical copies of the same triangle with sides \(a, b, c\) and (opposite) angles \(A, B, C\). Divide each into two right angled triangles.

\[
\begin{align*}
  x &= b \sin C \\
  x &= c \sin B \\
  \therefore b \sin C &= c \sin B \\
  \frac{\sin C}{c} &= \frac{\sin B}{b} \\

  y &= a \sin B \\
  x &= b \sin A \\
  \therefore a \sin B &= b \sin A \\
  \frac{\sin B}{b} &= \frac{\sin A}{a} \\

  z &= a \sin C \\
  z &= c \sin A \\
  \therefore a \sin C &= c \sin A \\
  \frac{\sin C}{c} &= \frac{\sin A}{a}
\end{align*}
\]

Hence the Sine Rule states

\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}
\]

or

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}
\]
To prove the **Cosine Rule**, consider three identical copies of the same triangle with sides a, b, c and (opposite) angles A, B, C. Divide each into two right angled triangles, as per the Sine Rule derivation. However, in this case we will combine some basic trigonometry with Pythagoras’ theorem to write two expressions for the dividing line x, y or z.

\[ b^2 = (b \cos A)^2 + y^2 \]
\[ a^2 = (c - b \cos A)^2 + y^2 \]
\[ : \cdot b^2 - (b \cos A)^2 = a^2 - (c - b \cos A)^2 \]
\[ b^2 - b^2 \cos^2 A = a^2 - \left\{ c^2 - 2bc \cos A + b^2 \cos^2 A \right\} \]
\[ b^2 - b^2 \cos^2 A = a^2 - c^2 + 2bc \cos A - b^2 \cos^2 A \]

\[ a^2 = b^2 + c^2 - 2bc \cos A \]
\[ A = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right) \]

\[ c^2 = (c \cos B)^2 + x^2 \]
\[ b^2 = (a - c \cos B)^2 + x^2 \]
\[ : \cdot c^2 - (c \cos B)^2 = b^2 - (a - c \cos B)^2 \]
\[ c^2 - c^2 \cos^2 B = b^2 - \left\{ a^2 - 2ac \cos B + c^2 \cos^2 B \right\} \]
\[ c^2 - c^2 \cos^2 B = b^2 - a^2 + 2ac \cos B - c^2 \cos^2 B \]

\[ b^2 = c^2 + a^2 - 2ac \cos B \]
\[ B = \cos^{-1}\left(\frac{c^2 + a^2 - b^2}{2ac}\right) \]

\[ a^2 = (a \cos C)^2 + z^2 \]
\[ c^2 = (b - a \cos C)^2 + z^2 \]
\[ : \cdot a^2 - (a \cos C)^2 = c^2 - (b - a \cos C)^2 \]
\[ a^2 - a^2 \cos^2 C = c^2 - \left\{ b^2 - 2ab \cos C + a^2 \cos^2 C \right\} \]
\[ a^2 - a^2 \cos^2 C = c^2 - b^2 + 2ab \cos B - a^2 \cos^2 C \]

\[ c^2 = a^2 + b^2 - 2ab \cos C \]
\[ C = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right) \]
For finding angles it is best to use the Cosine Rule, as cosine is single valued in the range $0^\circ$... $180^\circ$ whereas sine has two values. If the angle is obtuse (i.e. $>90^\circ$), then the sine rule can yield an incorrect answer since most calculators will only give the solution to $\sin \theta = k$ within the range $-90^\circ$ .... $90^\circ$
Surveying example:
Use the measurements below to work out the height $H$ of Mount Everest above sea level.

The method of determining altitude of mountain peaks using elevation measurements at either end of a flat baseline was used to great effect in the Great Trigonometrical Survey of India which continued for much of the 19th century. George Everest was the second superintendent of the GTS, and the world’s highest peak was (renamed) after him. In Nepal it is known as Sagarmatha, and in Tibet, Chomolungma “Mother Goddess of the Universe”.

Mathematics topic handout: Trigonometry – Sine rule and Cosine Rule

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