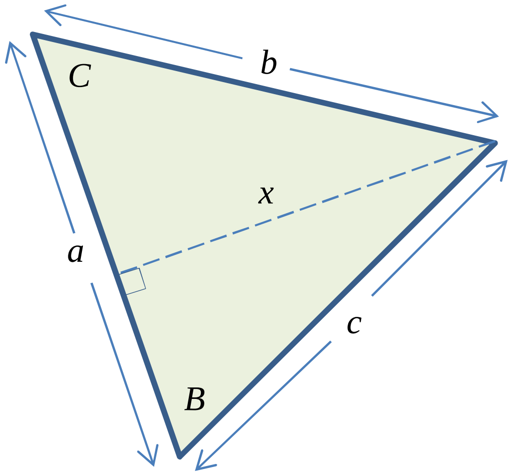
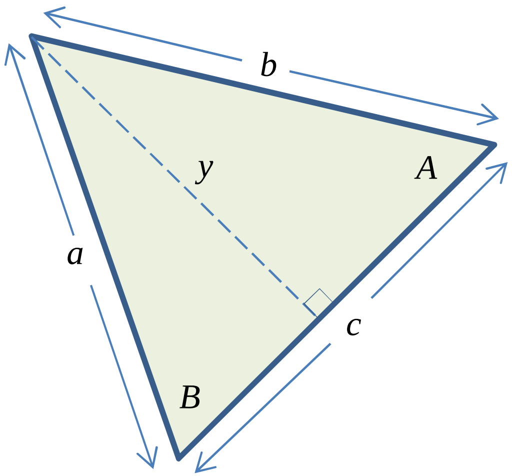
Sine Rule & Cosine Rule These are two extremely useful trignometric results which are applicable to *all* triangles, not just right angled ones.

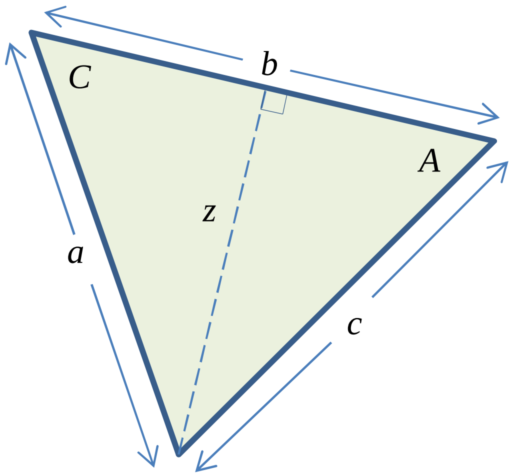
To prove the **Sine Rule**, consider three identical copies of the same triangle with sides a,b,c and (opposite) angles A,B,C.
Divide each into two right angled triangles.



$$x = b\sin C$$
$$x = c\sin B$$
$$\therefore b\sin C = c\sin B$$
$$\frac{\sin C}{c} = \frac{\sin B}{b}$$



$$y = a\sin B$$
$$x = b\sin A$$
$$\therefore a\sin B = b\sin A$$
$$\frac{\sin B}{b} = \frac{\sin A}{a}$$



$$z = a\sin C$$
$$z = c\sin A$$
$$\therefore a\sin C = c\sin A$$
$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

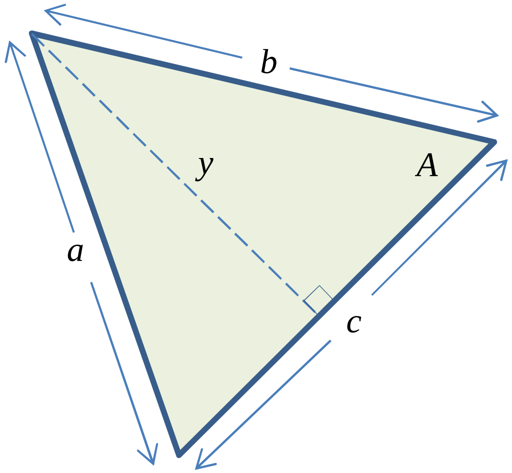
Hence the **Sine Rule** states

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

or

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

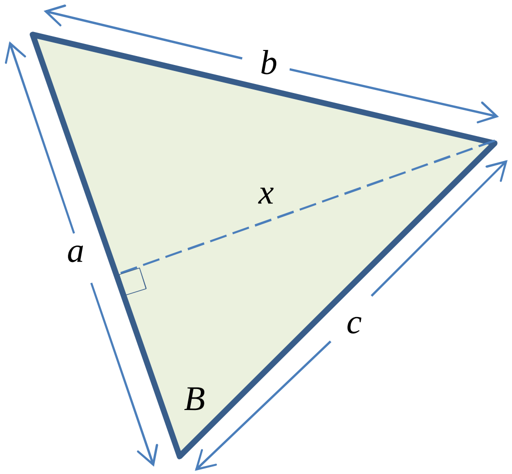
To prove the **Cosine Rule**, consider three identical copies of the same triangle with sides a,b,c and (opposite) angles A,B,C.
Divide each into two right angled triangles, as per the Sine Rule derivation. However, in this case we will combine some basic trigonometry with Pythagoras' theorem to write two expressions for the dividing line x, y or z.



$$b^2 = (b\cos A)^2 + y^2$$
$$a^2 = (c - b\cos A)^2 + y^2$$
$$\therefore b^2 - (b\cos A)^2 = a^2 - (c - b\cos A)^2$$
$$b^2 - b^2\cos^2 A = a^2 - \left\{c^2 - 2bc\cos A + b^2\cos^2 A\right\}$$
$$b^2 - b^2\cos^2 A = a^2 - c^2 + 2bc\cos A - b^2\cos^2 A$$

$$\boxed{a^2 = b^2 + c^2 - 2bc\cos A}$$

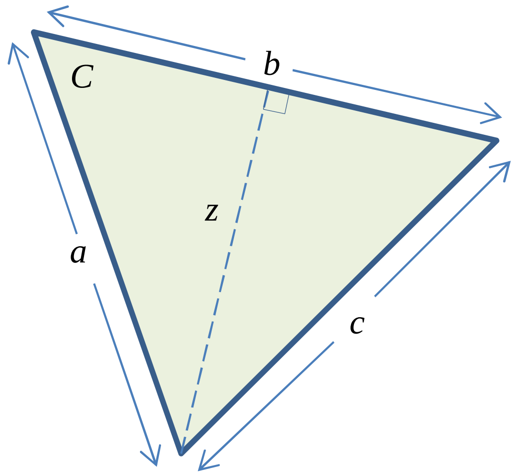
$$\boxed{A = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right)}$$



$$c^2 = (c\cos B)^2 + x^2$$
$$b^2 = (a - c\cos B)^2 + x^2$$
$$\therefore c^2 - (c\cos B)^2 = b^2 - (a - c\cos B)^2$$
$$c^2 - c^2\cos^2 B = b^2 - \left\{a^2 - 2ac\cos B + c^2\cos^2 B\right\}$$
$$c^2 - c^2\cos^2 B = b^2 - a^2 + 2ac\cos B - c^2\cos^2 B$$

$$\boxed{b^2 = c^2 + a^2 - 2ac\cos B}$$

$$\boxed{B = \cos^{-1}\left(\frac{c^2 + a^2 - b^2}{2ac}\right)}$$

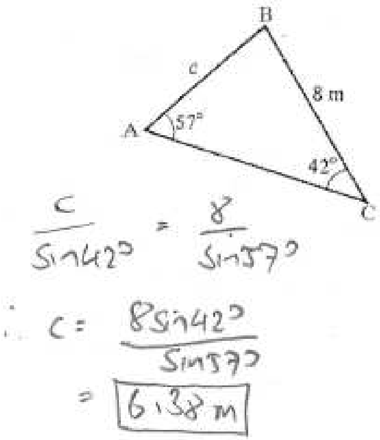


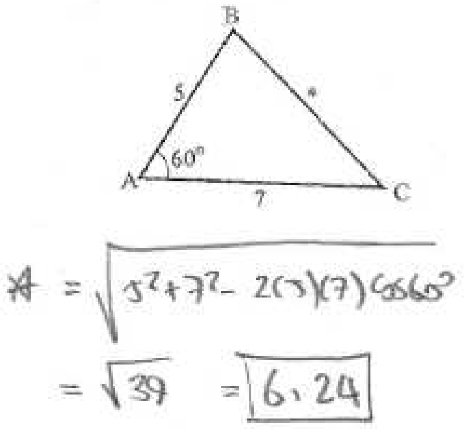
$$a^2 = (a\cos C)^2 + z^2$$
$$c^2 = (b - a\cos C)^2 + z^2$$
$$\therefore a^2 - (a\cos C)^2 = c^2 - (b - a\cos C)^2$$
$$a^2 - a^2\cos^2 C = c^2 - \left\{b^2 - 2ab\cos C + a^2\cos^2 C\right\}$$
$$a^2 - a^2\cos^2 C = c^2 - b^2 + 2ab\cos B - a^2\cos^2 C$$

$$\boxed{c^2 = a^2 + b^2 - 2ab\cos C}$$

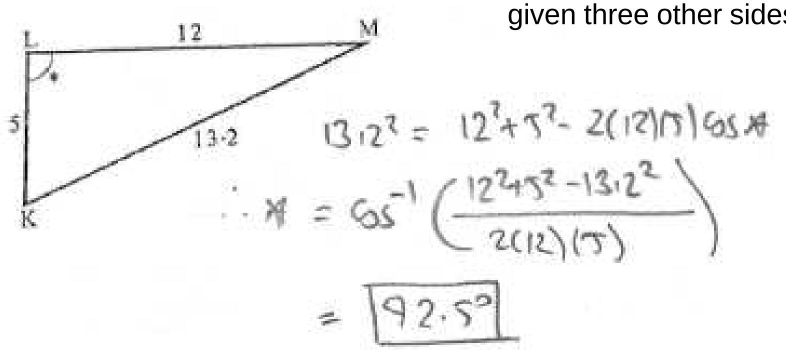
$$\boxed{C = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right)}$$

Examples:

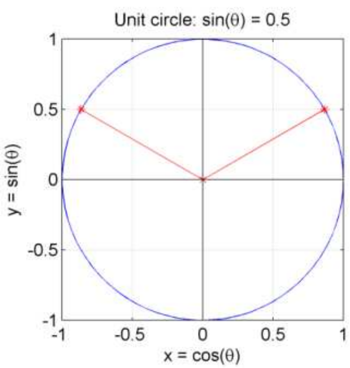


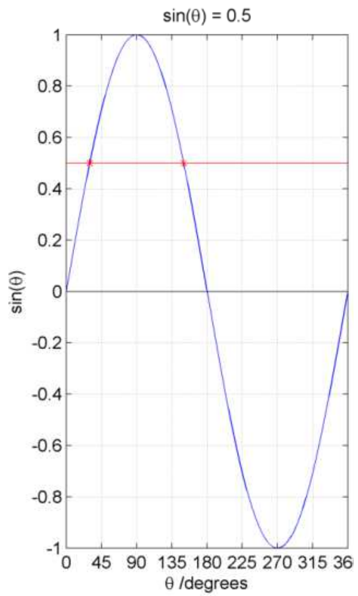


Use the cosine rule to find angles given three other sides



For *finding angles* it is best to use the *Cosine Rule*, as cosine is *single valued* in the range 0º... 180º whereas sine has *two* values.
If the angle is *obtuse* (i.e. > 90º), then the sine rule can yield an incorrect answer since most calculators will only give the solution to $\sin\theta = k$ within the range -90º 90º

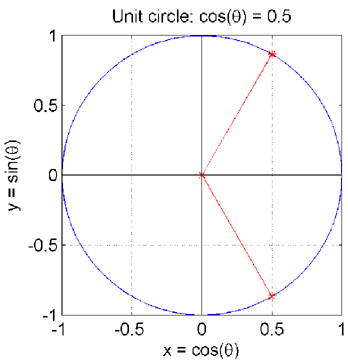


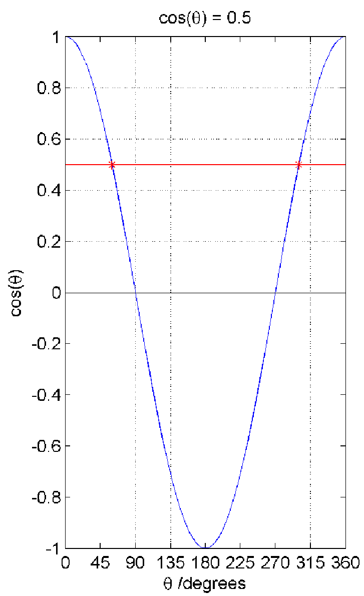


$\sin\theta = \frac{1}{2}$

$\theta = 30°, 150°, ...$

TWO values in the range 0º... 180º
Which one is correct?

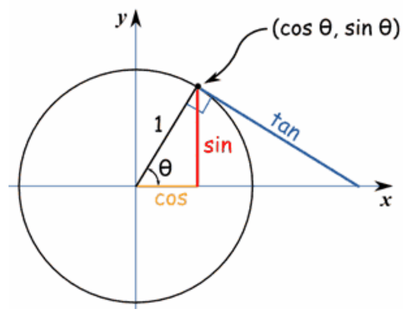




$\cos\theta = \frac{1}{2}$

$\theta = 60°, 300°, ...$

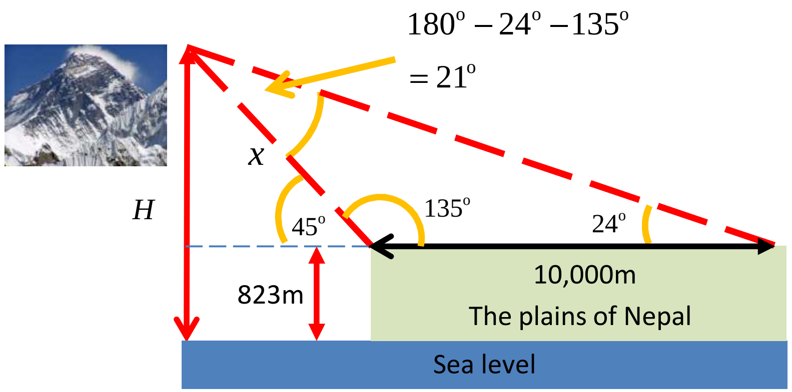
Only one solution in the range 0º... 180º



$\cos\theta$	**is the x coordinate of the unit circle**
$\sin\theta$	**is the y coordinate of the unit circle**
θ	**is measured anti-clockwise from the x axis**

Surveying example:

Use the measurements below to work out the height *H* of Mount Everest *above sea level*



Sine rule:

$$\frac{x}{\sin 24^\circ} = \frac{10{,}000}{\sin 21^\circ}$$

$$\therefore x = \frac{10{,}000 \sin 24^\circ}{\sin 21^\circ}$$

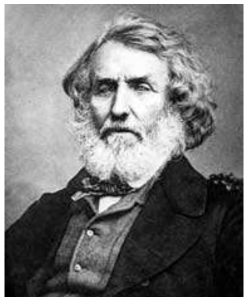
$$H - 823 = x \sin 45^\circ = \frac{x}{\sqrt{2}}$$

$$\therefore H = 823 + \frac{x}{\sqrt{2}}$$

$$\therefore H = 823 + \frac{10{,}000 \sin 24^\circ}{\sin 21^\circ \times \sqrt{2}}$$

$$\therefore \boxed{H = 8848\text{m}}$$

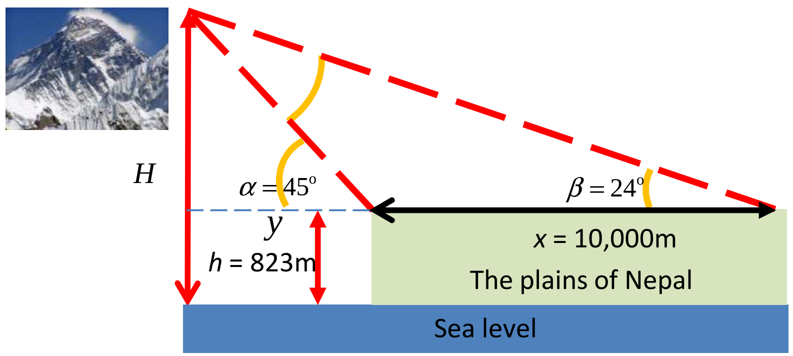
The method of determining altitude of mountain peaks using elevation measurements at either end of a flat baseline was used to great effect in the *Great Trigonometrical Survey of India* which continued for much of the 19^{th} century. George Everest was the second superintendent of the GTS, and the world's highest peak was (renamed) after him. In Nepal it is known as *Sagarmatha*, and in Tibet, *Chomolungma* "Mother Goddess of the Universe".



George Everest (1790-1866)

Surveying example (2):

Note the calculation can also be done more generally, and directly, using tan



$$H - h = (x + y)\tan\beta$$

$$H - h = y \tan\alpha$$

$$\therefore y = \frac{H - h}{\tan\alpha}$$

$$\therefore H - h = x\tan\beta + \frac{H - h}{\tan\alpha}\tan\beta$$

$$H\left(1 - \frac{\tan\beta}{\tan\alpha}\right) = h + x\tan\beta - h\frac{\tan\beta}{\tan\alpha}$$

$$H(\tan\alpha - \tan\beta) = h\tan\alpha + x\tan\alpha\tan\beta - h\tan\beta$$

$$\boxed{H = h + \frac{x\tan\alpha\tan\beta}{\tan\alpha - \tan\beta}}$$

$$\therefore H = 823 + \frac{10{,}000 \tan 45^\circ \tan 24^\circ}{\tan 45^\circ - \tan 24^\circ}$$

$$\therefore \boxed{H = 8848\text{m}}$$