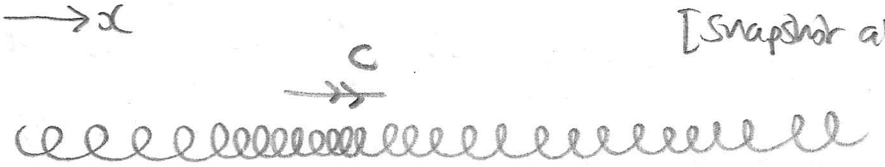
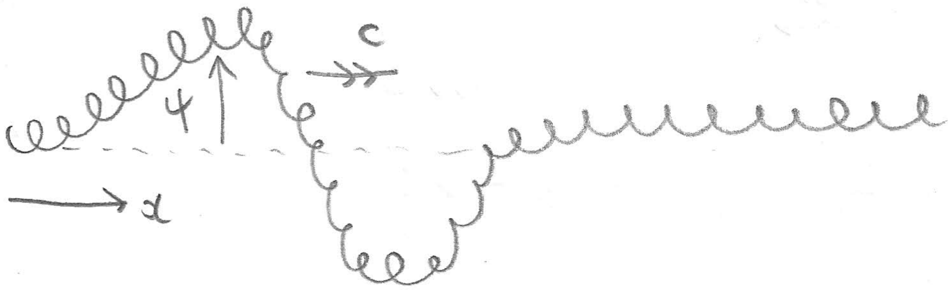
WAVES

[Snapshot at time t]

Q1/ (i)



- Longitudinal wave on a slinky
- $\psi(x,t)$ is the compression of the slinky from the equilibrium separation of the coils.
- The direction of compression is $\parallel$ to the propagation direction of the wave. c is the wave speed.



- TRANSVERSE wave on a slinky
- $\psi(x,t)$ is the $\perp$ displacement to the main axis of the slinky, and the direction of wave propagation.

(ii) E string: $f = 82.41\,\text{Hz}$. $c = 340\,\text{m/s}$

b) $\boxed{c = f\lambda}$ $\therefore$ $\lambda = c/f$

$\lambda = 340/82.41 = \boxed{4.13\,\text{m}}$

a) $T = \frac{1}{f} = \frac{1}{82.41} = \boxed{1.21 \times 10^{-2}\,\text{s}}$ (or 12.1 ms)

c) $k = 2\pi/\lambda = 2\pi/(340/82.41) = \boxed{1.52\,\text{m}^{-1}}$

d) $\omega = 2\pi f = 2\pi \times 82.41 = \boxed{517.8\,\text{rad s}^{-1}}$

check that $\boxed{\omega = ck}$: $\omega/k = \frac{517.8}{1.52} = 340.7\,\text{m/s}$ ✓

(involves rounding error!)

(iii) $\boxed{\psi(x,t) = 3\cos(2.2x - 55t)} = A\cos(kx - \omega t)$

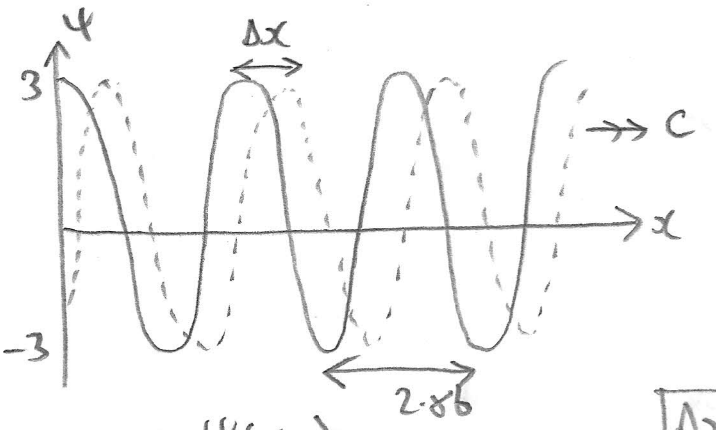
a) Amplitude $A = 3$. $\quad k = 2.2. \quad \omega = 55$.

b) $k = 2\pi/\lambda \quad \therefore \quad \frac{2\pi}{\lambda} = 2.2 \Rightarrow \lambda = 2\pi/2.2 = \boxed{2.86}$

c) $\omega = 2\pi f \quad \therefore f = \omega/2\pi = \frac{55}{2\pi} = \boxed{8.75}$

d) $T = 1/f = \frac{2\pi}{55} = \boxed{0.11}$

{ No units given but assume wavelength in m frequency in Hz period in s }



— $\psi(x,0)$

--- $\psi(x,\Delta t)$

$\boxed{\Delta x = c\Delta t}$ ie translation of $\psi(x,0)$ by Δx.

$\psi(x,t) = 3\cos(2.2x - 55t)$

$\Delta t \ll T$.

(iv) Ultrasound: $\lambda = c/f = \frac{1500}{50\times10^3} = \boxed{0.03\text{ m}}$

Microwaves: $\lambda = c/f = \frac{2.998\times10^8}{10\times10^9} = \boxed{0.03\text{ m}}$

ie the same wavelength. This is why SONAR antennae underwater are $\approx$ the same size as radar antennae (which work at frequencies 1 – 10 GHz).

(v) $P = \frac{1}{2}ZA^2\omega^2$ wave power. (Z is "wave impedance")

ie $P \propto A^2 f^2$ since $\omega = 2\pi f$

$\therefore 2A_2^2 f_2^2 = A_4^2 f_4^2 \quad \therefore \frac{A_2}{A_4} = \frac{f_4}{f_2}\frac{1}{\sqrt{2}} = \frac{104.9}{88.1\sqrt{2}}$

(2) (R4 broadcast at twice the power)

$= \boxed{0.842}$

Alternatively, $\frac{A_4}{A_2} = \boxed{1.14}$

(vi) $\boxed{c = \sqrt{\gamma P/\rho}}$ $\therefore$ $c^2 = \gamma P/\rho$ $\therefore$ $\boxed{\rho = \frac{\gamma P}{c^2}}$

$\therefore \rho = \frac{1.40 \times 101325}{340^2} = \boxed{1.23 \text{ kg/m}^3}$ {Density of air}

(vii) $\Delta t = \Delta t_w + \Delta t_g + \Delta t_c$

wood glass concrete

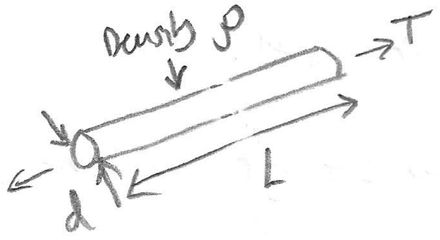
$= \frac{x_w}{c_w} + \frac{x_g}{c_g} + \frac{x_c}{c_c}$ $\boxed{c = \sqrt{K/\rho}}$

$= \frac{2.0}{\sqrt{\frac{28.3 \times 10^9}{700}}} + \frac{3.0}{\sqrt{\frac{105 \times 10^9}{2500}}} + \frac{1.5}{\sqrt{\frac{58 \times 10^9}{2400}}}$ (s)

$= \boxed{1.08 \times 10^{-3} \text{ s}}$ (or 1.08 ms)

Speeds are: (in m/s)

wood	6358
glass	6481
concrete	4916

(viii) 

(Guitar string)

String under tension T, waves are of speed $c = \sqrt{T/\mu}$

Mass of string is $M = \pi \left(\frac{d}{2}\right)^2 L \rho$

$\therefore \mu = \frac{M}{L} = \frac{\pi d^2 \rho}{4}$

Now for fundamental: $L = \lambda/2$ $c = f\lambda$ $\therefore$ $L = \frac{c}{2f}$

$\therefore \quad L^2 = \frac{c^2}{4f^2}. \qquad c^2 = \frac{T}{\mu} = \frac{4T}{\pi d^2 \rho}$

$\therefore \quad L^2 = \frac{4T}{\pi d^2 \rho} \frac{1}{4f^2} = \frac{T}{\pi d^2 \rho f^2}$

$\therefore \quad \boxed{L = \sqrt{\frac{T}{\pi \rho}} \frac{1}{df}}$

$\therefore \quad L = \sqrt{\frac{94.3}{\pi \times 6610}} \times \frac{1}{0.036 \times 0.0254 \times 110} \quad (m)$

$\boxed{L = 0.700 \text{ m}}$

Now $\quad \boxed{T = \pi \rho (Ldf)^2}$

So if $f = \frac{110}{\sqrt[12]{2}}$ and everything else is the same

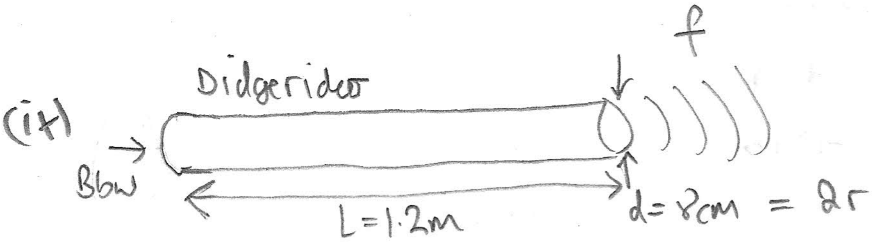
ie $A \rightarrow Ab$ ("A flat")

$T \rightarrow T \times (2^{-\frac{1}{12}})^2$

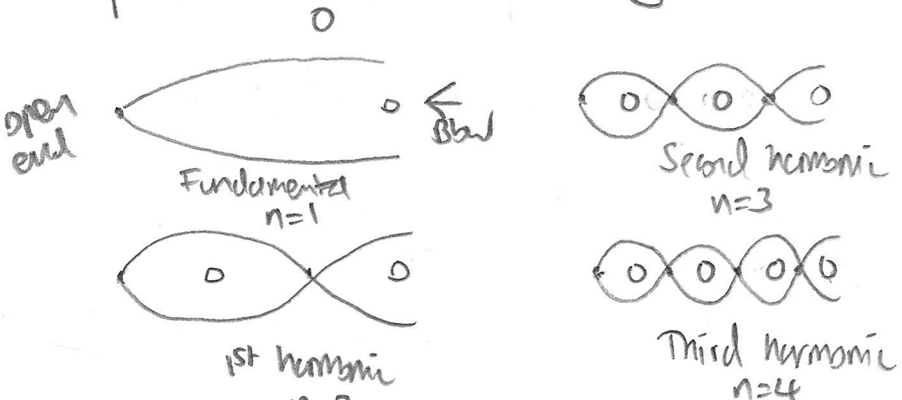
$T \rightarrow T \times 2^{-\frac{1}{6}} \qquad$ ie $\quad T = 94.3N \times \frac{1}{1.122}$

$= \boxed{84.0 \text{ N}}$

(ii)



open ended tube so $\therefore$ pressure node at open end and pressure antinode and blowing end.



So $\underbrace{L + \frac{2}{3}r}_{\text{end correction}} = (2n-1)\frac{\lambda_n}{4}$

Now $\quad c = f_n \lambda_n$

$\therefore \quad L + \frac{2}{3}r = (2n-1)\frac{c}{4f_n}$

$\therefore \quad \boxed{f_n = \frac{(2n-1)c}{4(L + \frac{2}{3}r)}}$

So if $c = 340\,M/s$, $L = 1.2M$, $r = 0.04M$

$$f_n/Hz = \frac{(2n-1)\times 340}{4(1.2 + \frac{2}{3}\times 0.04)}$$

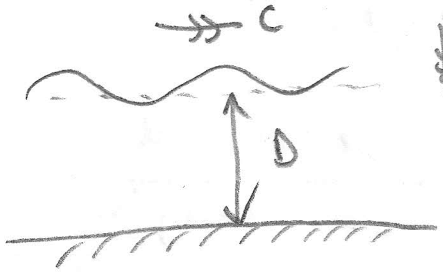
$$\boxed{f/Hz = (2n-1)\times 69.29}$$

So

n	f_n / Hz
1	69.29
2	207.88
3	346.47
4	485.05

(x) Shallow water waves, where wave amplitude is large enough for ripples to be ignored, and ∴ surface tension.

So dispersion relationship is: $\boxed{\omega^2 = gk^2 D}$



$g = 9.81\ N/kg$ ∴ $\omega = k\sqrt{gD}$

∴ $c_p = \frac{\omega}{k} = \boxed{\sqrt{gD}}$ phase velocity

Towards shore: $D = 20.0M$, $A = 2.0M$

$$c_p = \sqrt{9.81 \times 20.0} = \boxed{14.0\ M/s}$$

Reef: $D = 3.0M$. $c_p = \sqrt{9.81\times 3.0}$

$$= \boxed{5.42\,M/s}$$

Now if the amount of water moving remains the same (water is not very compressible so this is a good approximation)

Ac_p = constant ∴ $2.0 \times 14.0 = A_{reef} \times 5.42$

∴ $A_{reef} = \boxed{5.16M}$ (is a pretty huge wave!)

Q2/ Alpine horn "labiophone"

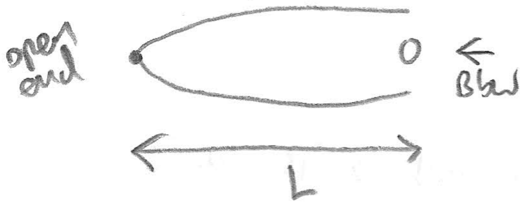
"Blow" end must be an antinode of pressure o

open end must be a node of pressure (since at atmospheric pressure) •

a)

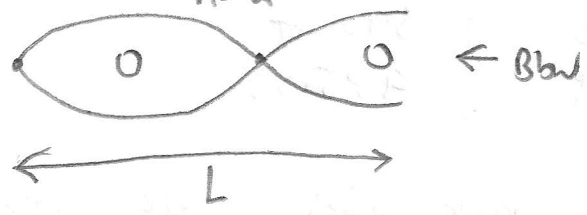
So possible standing waves are:

$n=1$



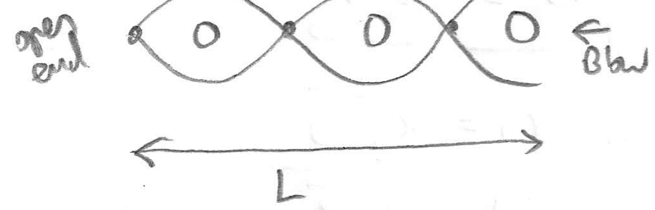
Fundamental

$n=2$



1st harmonic

$n=3$



2nd harmonic

ie L = odd # of quarter wavelengths

so if we ignore the end correction : $\boxed{L = (2n-1)\frac{\lambda_n}{4}}$

b) Now $c = f_n \lambda_n \therefore \lambda_n = c/f_n$

$\therefore \quad L = (2n-1)\, c/4f_n \quad \Rightarrow \quad \boxed{f_n = (2n-1)\, c/4L}$

So $f_n = (2n-1) \times 342/4 \times 2.4 = \boxed{35.6\text{ Hz}} \times (2n-1)$

n	f/Hz
1	35.6
2	106.9
3	178.1
4	249.4
5	320.6

n	f/Hz
6	391.9
7	463.1
8	534.4
9	605.6

$\boxed{\text{Equi-tempered Scale}}$: $f_m = 440\,\text{Hz} \times 2^{m/12}$ $(m \in \mathbb{Z})$

m	f/Hz	Note
0	440	A
1	466.2	Bb
2	493.9	B
3	523.3	C
4	554.4	C#
5	587.3	D
6	622.3	Eb

For alpine horn:

So: $f_7 = 463.1$ Hz is slightly flat of Bb

$f_8 = 534.4$ Hz is between a C and C#

$f_9 = 605.6$ Hz is between a D and Eb

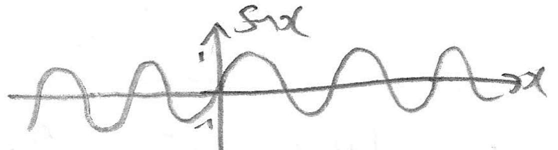
c) what if n=7 harmonic was exactly 440Hz? (i.e. A exactly, not 463.1 Hz)

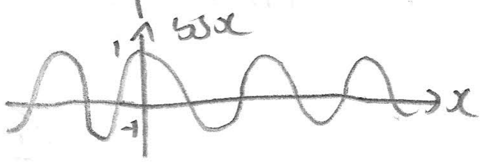
$$\therefore \quad 440 = (2\times7-1)\times\frac{c}{4L}$$

$$\therefore \quad L = \frac{13\times342}{4\times440} = \boxed{2.53\,\text{m}}$$

Hence if $L = 2.53$ m, $f_n = (2n-1)\frac{c}{4L}$

$$\boxed{f_n = 33.85\,\text{Hz} \times (2n-1)}$$





Q3/ $\psi = A\sin(kx-\omega t) - A\sin(-kx-\omega t)$

$= A\sin kx\cos\omega t + A\cos kx\sin\omega t$

$- \{A\sin(-kx)\cos\omega t + A\cos(-kx)\sin\omega t\}$

Now $\sin(-kx) = -\sin kx$ and $\cos(-kx) = \cos kx$

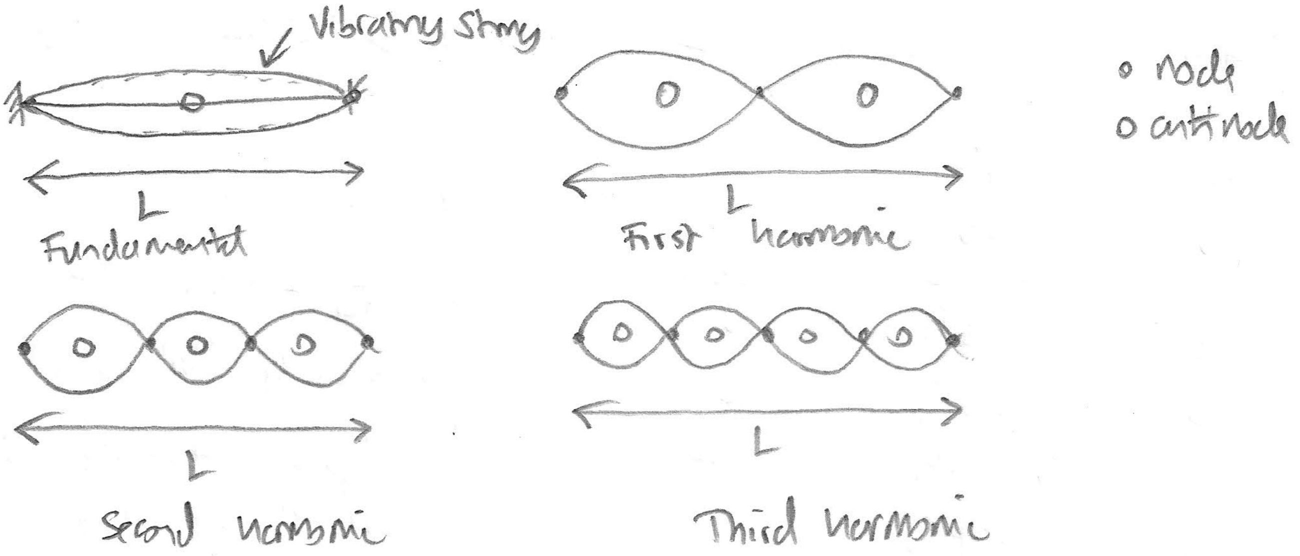
Since sine is an odd function and cosine is even.

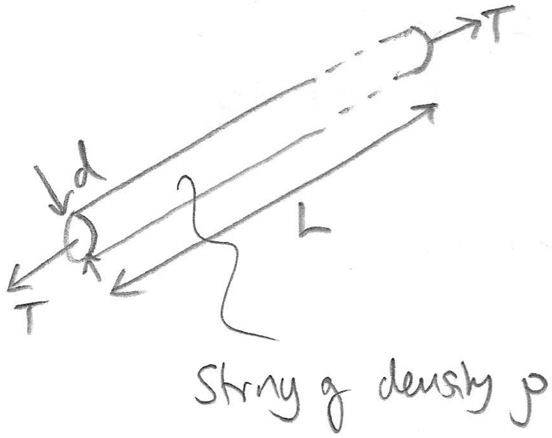
[$f(-x) = -f(x) \Rightarrow$ ODD ; $f(-x) = f(x) \Rightarrow$ EVEN]

$\therefore \psi = A\{\sin kx\cos\omega t + \cancel{\cos kx\sin\omega t} + A\sin kx\cos\omega t - \cancel{A\cos kx\sin\omega t}\}$

$$\therefore \boxed{\psi = 2A\sin kx\cos\omega t}$$

Q4/





Wave on guitar string under tension T is $\boxed{c = \sqrt{T/\mu}}$

Mass per unit length:

$$\mu = \frac{m}{L} = \frac{\rho \pi (d/2)^2 L}{L}$$

$$\Rightarrow \boxed{\mu = \rho \pi d^2/4}$$

$$\therefore \quad T = \mu c^2$$

$$\boxed{T = \frac{\rho \pi d^2}{4} c^2}$$

Now $\boxed{c = f_n \lambda_n}$ and for standing waves:

$n \frac{\lambda_n}{2} = L$ ie L = integer # of half wavelengths.

so $\lambda_n = 2L/n$ let $\lambda = \lambda_1 = 2L$.

$$\therefore \quad \boxed{c = 2Lf}$$

$$\therefore \quad T = \frac{\rho \pi d^2}{4} 4 L^2 f^2 \Rightarrow \boxed{T = \rho \pi (f L d)^2}$$

For Fender Precision Bass : $L = 0.864$ m and bottom E string (41.204 Hz) has a diameter of 2.67 mm.

if $\rho = 7850$ kg/m3 :

$$\Rightarrow T = 7850 \times \pi \left(41.204 \times 0.864 \times 2.67 \times 10^{-3}\right)^2$$

$$= \boxed{222.8\ N}$$

{ ie a mass of 22.7 kg has weight = 222.8 N }

So if T is the same for all four strings, and these are made from steel of $\rho = 7850$ kg/m³

$$\sqrt{\frac{T}{\rho\pi}} = fLd \quad \therefore \quad \boxed{d = \frac{1}{fL}\sqrt{\frac{T}{\rho\pi}}}$$

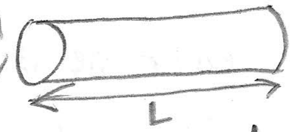
$$\therefore \ d = \frac{1}{f} \times \frac{1}{0.864}\sqrt{\frac{222.8}{7850 \times \pi}} \times 1000 \quad \text{(mm)}$$

$$\boxed{d = \frac{110.01}{f/\text{Hz}} \ \text{mm}}$$

$\therefore$

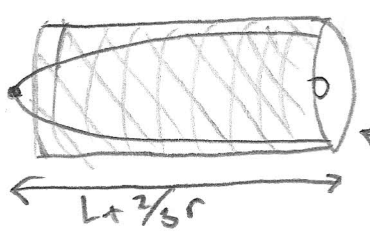
String notes	f/Hz	d/mm
G	97.999	1.10
D	73.416	1.50
A	55.000	2.00
E	41.204	2.67

Q5/ Diaphone-Dulzian pipe in Atlantic City. 2r



Fundamental

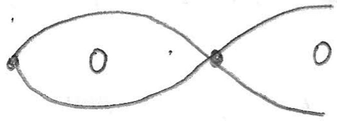
open end



← Driven end of pipe

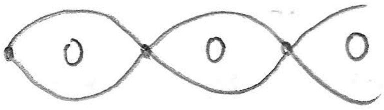
The wave represents air pressure change from atmospheric, it is LONGITUDINAL, not transverse!

open end

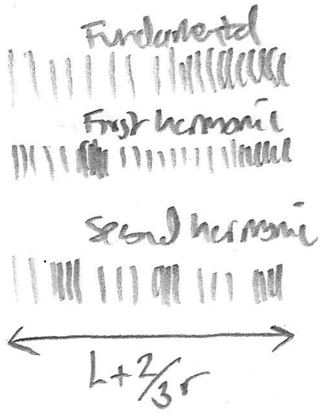


First harmonic

open end



Second harmonic



so $L+\frac{2}{3}r = (2n-1)\lambda_n/4$

to meet the condition of o antinode of pressure at driven end and • node of pressure at open end.

$c = f_n \lambda_n$ $\therefore$ $L+\frac{2}{3}r = (2n-1)\frac{c}{4f_n}$

$$\therefore \boxed{L = (2n-1)\frac{c}{4f_n} - \frac{2}{3}r}$$

$$\text{so} \quad \boxed{f_n = \frac{(2n-1)c}{4(L+\frac{2}{3}r)}}$$

$\pi r^2 = 1296 \times (0.0254)^2 \quad m^2$ (1296 square inches)

$\therefore$ pipe radius is $r = \sqrt{\frac{1296 \times (0.0254)^2}{\pi}}$

$\boxed{r = 0.516m}$ (!!)

so $f_n = \frac{(2n-1) \times 340}{4(19.74 + \frac{2}{3} \times 0.516)}$ (Hz)

(15)

$$\therefore \boxed{f_n = (2n-1) \times 4.23\ \text{Hz}}$$

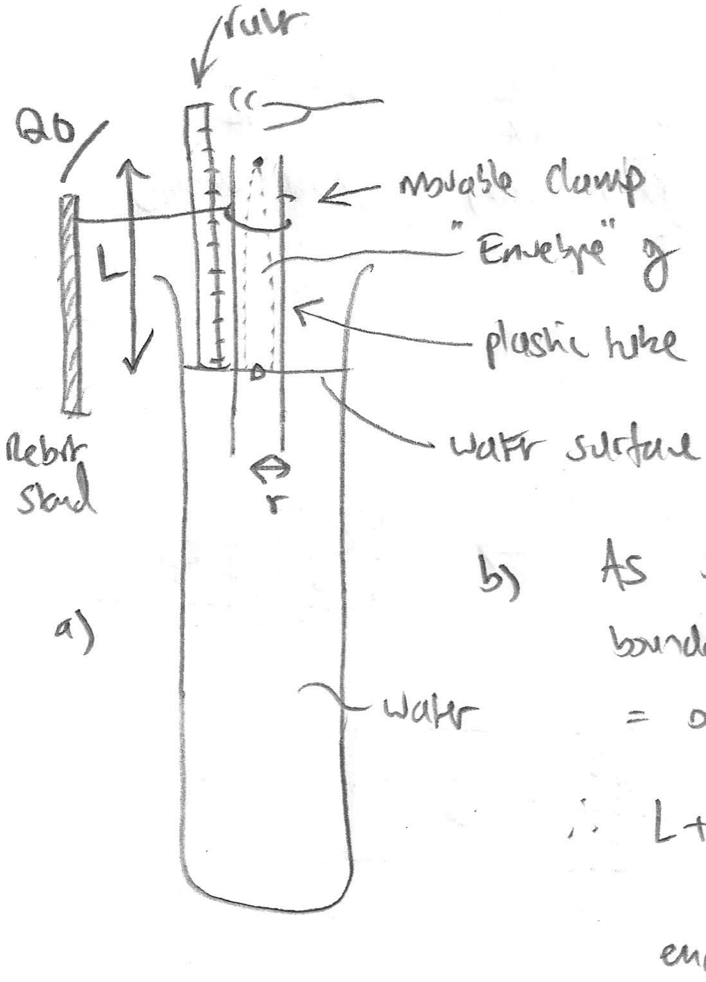
So in principle, the Diaphone-Dulzian pipe should be able to play CCCCC (4.09 Hz) if the speed of sound was slightly lower than 340 m/s in New Jersey!

The other harmonics are:

n	f/Hz
1	4.23
2	12.70
3	21.16
4	29.63
5	38.09

Strangely, on the 'web' it states the lowest note is 8 Hz, not 4 Hz. This would imply a 32 ft pipe not, 64 ft .... or perhaps there is something about organ pipes which permits extra harmonics?

Q6/



a)

b) As shown above, • ..... o boundary conditions $\Rightarrow$ $L + \frac{2}{3}r$ = odd # of quarter wavelengths.

$$\therefore \underbrace{L + \tfrac{2}{3}r}_{\text{end correction}} = (2n-1)\lambda_n/4$$

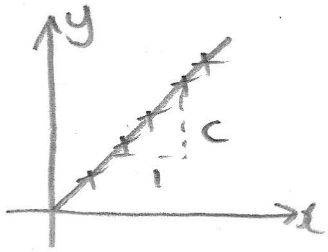
$$c = f_n \lambda_n$$

So $L + \frac{2}{3}r = \frac{c}{4f_n}(2n-1)$

$\therefore$ If f is the tuning fork frequency, and the first resonance is heard when $f_n = f$

$$\therefore \boxed{L = \frac{c}{4f} - \frac{2}{3}r}$$

c) plot $y = L + \frac{2}{3}r$ vs $x = \frac{1}{4f}$ and you should get a graph $y = cx$ ie with gradient of the speed of sound, that passes through the origin.



See spreadsheet: $\boxed{c \simeq 347 \text{ m/s}}$

Q7/ a) Water waves: $c_g = \frac{d\omega}{dk}$ group velocity $c_p = \frac{\omega}{k}$ phase velocity

$\boxed{\text{Deep water}}$: $\omega^2 = gk$

so $\omega = \sqrt{g}\, k^{\frac{1}{2}}$

$\therefore$ $c_g = \frac{d\omega}{dk} = \boxed{\frac{1}{2}\sqrt{g}\, k^{-\frac{1}{2}}}$

$c_p = \frac{\omega}{k} = \boxed{\sqrt{g}\, k^{-\frac{1}{2}}}$

$\Big\} \rightarrow \therefore \boxed{c_g = \frac{1}{2}c_p}$

b) Deep water ripples: $\boxed{\omega^2 = \frac{\sigma k^3}{\rho} + gk}$

(ie $kD \gg 1$

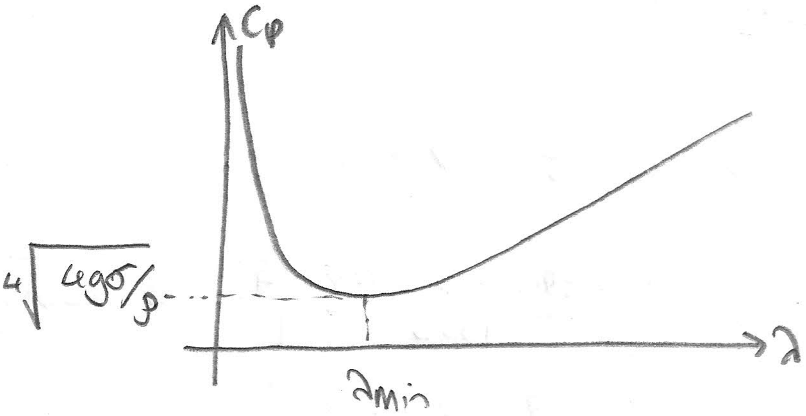
$\Rightarrow \frac{2\pi}{\lambda}D \gg \Rightarrow 2\pi D \gg \lambda$

So deep water)

$$\therefore \quad \frac{\omega^2}{k^2} = \frac{\sigma k}{\rho} + g/k \qquad (c_p = \omega/k)$$

$$\therefore \quad c_p = \sqrt{\frac{\sigma k}{\rho} + g/k} \qquad (k = 2\pi/\lambda)$$

$$\therefore \quad \boxed{c_p = \sqrt{\frac{2\pi\sigma}{\lambda\rho} + g\lambda/2\pi}}$$



$$c_p^2 = \frac{2\pi\sigma}{\lambda\rho} + g\lambda/2\pi$$

$$2c_p \frac{dc_p}{d\lambda} = -\frac{2\pi\sigma}{\lambda^2\rho} + g/2\pi$$

so $\frac{dc_p}{d\lambda} = 0$ when $g/2\pi = \frac{2\pi\sigma}{\lambda^2\rho}$

$$\therefore \quad \lambda^2 = 4\pi^2\sigma/\rho g$$

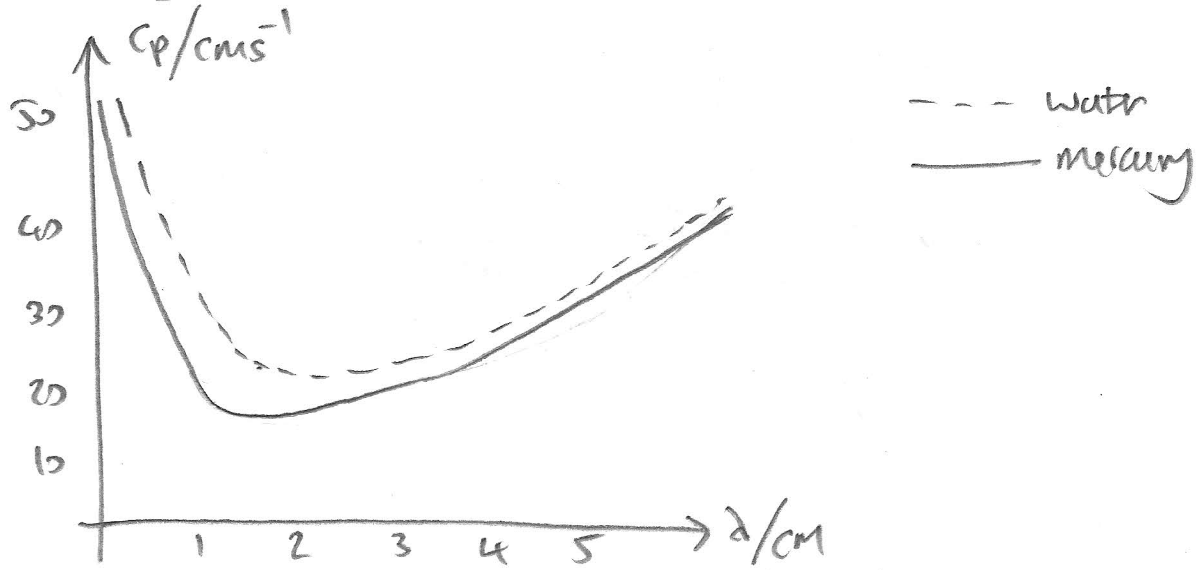
$$\therefore \quad \boxed{\lambda_{min} = 2\pi\sqrt{\sigma/\rho g}}$$

$\therefore$ c_p minima @ $c_p = \sqrt{\frac{2\pi\sigma}{\rho}\frac{\sqrt{\rho g}}{2\pi\sqrt{\sigma}} + \frac{g}{2\pi} \times 2\pi\sqrt{\sigma/\rho g}}$

$$c_p = \sqrt{\sqrt{\sigma g/\rho} + \sqrt{\sigma g/\rho}}$$

$$\boxed{c_p = \left(\frac{4\sigma g}{\rho}\right)^{1/4}}$$

See spreadsheet for accurate plot of C_p vs λ for water and mercury:



Min. C_p :
(Minimum phase velocity)

Water : $\left(\frac{4\times9.81\times0.0728}{1000}\right)^{\frac{1}{4}} = \boxed{0.0231\,\mathrm{m/s}}$

Mercury : $\left(\frac{4\times9.81\times0.487}{13600}\right)^{\frac{1}{4}} = \boxed{0.0194\,\mathrm{m/s}}$

Note as $\lambda \to$, $C_p \to \sqrt{\frac{g\lambda}{2\pi}}$ which will be the same C_p vs $\sqrt{\lambda}$ for both water and mercury. (until deep water $kD \gg 1$ approximation becomes invalid).

Q8/ Wave equation : $\boxed{\frac{\partial^2\psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}}$

a) $\boxed{\psi(x,t) = A\cos(kx - \omega t - \phi)}$

$\frac{\partial\psi}{\partial x} = -Ak\sin(kx-\omega t-\phi)$ $\qquad$ $\frac{\partial\psi}{\partial t} = A\omega\sin(kx-\omega t-\phi)$

$\frac{\partial^2\psi}{\partial x^2} = -Ak^2\cos(kx-\omega t-\phi)$ $\qquad$ $\frac{\partial^2\psi}{\partial t^2} = -A\omega^2\cos(kx-\omega t-\phi)$

Now $\omega = ck$ $\therefore$ $\omega^2 = c^2k^2$

So: $\frac{\partial^2\psi}{\partial x^2} = -k^2\psi$ $\qquad$ $\frac{\partial^2\psi}{\partial t^2} = -c^2k^2\psi$ $\quad\therefore$ $\boxed{\frac{\partial^2\psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}}$

b) $\boxed{\psi(x,t) = 2A \sin kx \cos \omega t}$

$$\frac{\partial^2 \psi}{\partial x^2} = -k^2 \psi \qquad \frac{\partial^2 \psi}{\partial t^2} = -\omega^2 \psi$$

i.e. same as (a) so $\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$ ✓

c) $\boxed{\psi(x,t) = Ae^{i(kx - \omega t - \phi)}}$

$$\frac{\partial \psi}{\partial x} = ik\, Ae^{i(kx-\omega t-\phi)}$$

$$\frac{\partial^2 \psi}{\partial x^2} = -k^2 \psi \qquad (i^2 = -1)$$

$$\frac{\partial \psi}{\partial t} = -i\omega\, Ae^{i(kx-\omega t-\phi)}$$

$$\frac{\partial^2 \psi}{\partial t^2} = i^2 \omega^2 Ae^{i(kx-\omega t-\phi)} = -\omega^2 \psi$$

So since $\omega = ck$ $\Rightarrow$ $\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$ ✓

AF 25/7/20.