A *wave* is essentially a disturbance of quantity $\psi(x,t)$ that propagates spatially at a fixed velocity c. The disturbance could be a change in gas pressure, movement of a string under tension, ground or water movement, or indeed fluctuations in electromagnetic fields which constitute light, radio waves, X-rays etc.

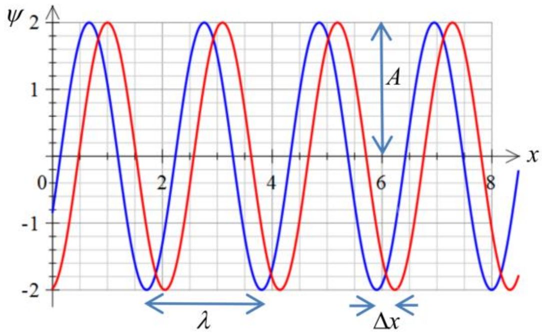
Longitudinal waves correspond to disturbances in the *same direction* as the wave propagation. e.g. the *compressions* and *rarefactions* of air pressure in sound waves or earthquake P-waves.

Transverse waves correspond to disturbances that are *perpendicular* to the wave propagation direction. e.g. electric and magnetic field directions in light waves, or ground movement in earthquake shear (S) waves.

Surface waves can occur between the boundaries of different materials. Water waves, earthquake Rayleigh (vertical motion) and Love (side-to-side i.e. shear) waves are classic examples. Wave patterns also can form from instabilities in layers of moving fluid of different densities, velocities and viscosities. The 'mare's tails'(this is called the Kelvin-Helmholtz instability) that form on aircraft contrails are everyday examples.

A wave[1] $\psi(x,t)$ is a *spatial translation of a disturbance* $f(x)$ as time progresses. $\psi(x,t) = f(x - ct)$. This means all waves obey the **wave equation**: $\partial^2\psi/\partial x^2 = \frac{1}{c^2}\partial^2\psi/\partial t^2$.

Periodic waves will *repeat spatially* with **wavelength** λ. We could measure this distance if we were to take a photograph of a wave at a given time. Alternatively, we could measure $\psi(x,t)$ at a *fixed position* x and record the time T for the wave to repeat. This is called the **period**. The number of waves per second is the **frequency** $\boxed{f = 1/T}$. Hence since wave speed $c = \lambda/T \Rightarrow \boxed{c = f\lambda}$.



Since all waveforms can be constructed from a sum of sine and cosine waves of different *amplitude*, *frequency* and *phase*,[2] we can use a waveform of the form $\psi(x,t) = A\cos\left(2\pi\frac{(x-ct)}{\lambda} - \phi\right)$ to represent the basic components of all waves. The **phase**[3] of the wave is $2\pi\frac{(x-ct)}{\lambda} - \phi$, and therefore ϕ is a *fixed phase shift* that may be required to describe waves of the same frequency, but are not 'in-phase' (e.g. in *diffraction* or *interference* contexts). A is the **amplitude** of the wave.

Two useful simplifications: **wavenumber**: $\boxed{k = 2\pi/\lambda}$ and **angular frequency** $\boxed{\omega = 2\pi/T = 2\pi f}$.

Hence: $\boxed{\psi(x,t) = A\cos(kx - \omega t - \phi)}$ and $\boxed{\omega = ck}$.

Note in many situations it is convenient to use *complex numbers* to represent wave phenomena. *De-Moivre's theorem* $e^{i\theta} = \cos\theta + i\sin\theta$ can be employed to write $\boxed{\psi(x,t) = Ae^{i(kx-\omega t-\phi)}}$. Take the real part to get the 'real' wave! This is very useful in a mathematical sense as exponentials differentiate and integrate to form the same exponentials (but scaled by a constant), whereas sine and cosine functions interchange between each other. i.e. $\frac{d}{dx}\left(e^{ikx}\right) = ike^{ikx}$, $\frac{d}{dx}(\cos kx) = -k\sin kx$

When considering waves moving in two or three dimensions, the ***wavevector*** $\mathbf{k}$ defines the direction of propagation. The magnitude of the wavevector is the wavenumber. If $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ is a *position vector* (*x*,*y*,*z*) from some origin:

$$\boxed{\psi(x,y,z,t) = \psi(\mathbf{r},t) = A\cos(\mathbf{k}\cdot\mathbf{r} - \omega t - \phi)}$$

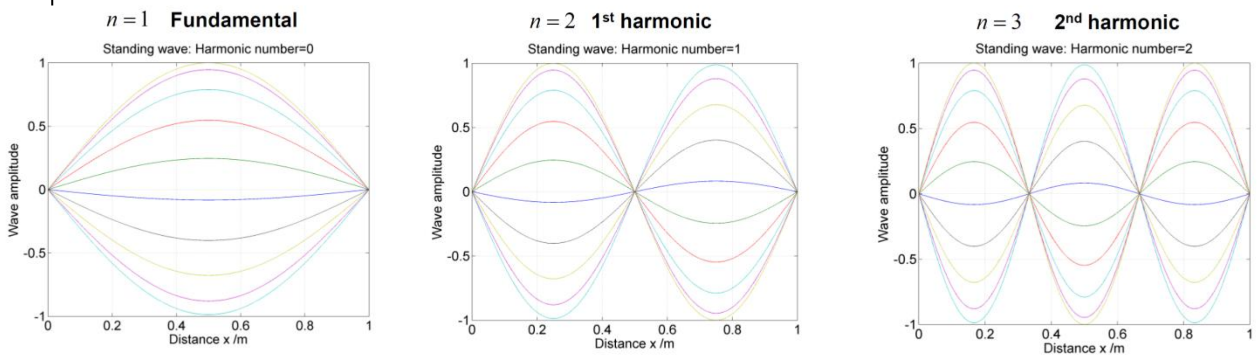
[1] We will use one-dimensional examples, but the ideas generalize to 2D or 3D.
[2] i.e. a *Fourier Series*
[3] The *phase*, i.e. the input of a sine or cosine function, or 'where you are within a wave', as defined, is in *radians*.

Standing waves can form when the geometry of the wave-system constrains *nodes* and *antinodes* to be **fixed** spatially. A **node** *is a point of zero disturbance*, whereas an **antinode** *is a point of maximum amplitude.* Standing waves form from a **superposition** (i.e. 'signed addition') of *incoming* and *reflected waves*, such that the standing wave's mathematical form is a *product* of temporal and spatial oscillations. This explains why the nodes and antinodes are fixed spatially. The entire standing wave still oscillates at frequency f, but the effective amplitude of these vibrations varies in a sinusoidal fashion along the length of the oscillating system.

Note waves on a string will be *inverted* upon reflection off a fixed end.

$$\psi(x,t) = \underbrace{A\sin(kx-\omega t)}_{\text{incoming wave}} - \underbrace{A\sin(-kx-\omega t)}_{\text{reflected wave}} \;\therefore\; \boxed{\psi(x,t) = 2A\sin kx\cos\omega t}$$

The fixed positions of nodes or antinodes will *quantize* the allowed frequencies of oscillation of standing waves. A tensioned string *must have nodes at both ends,* and any integer number of antinodes in between. (These are called 'harmonics'). **This means the length of the string must be an integer number of half-wavelengths.**



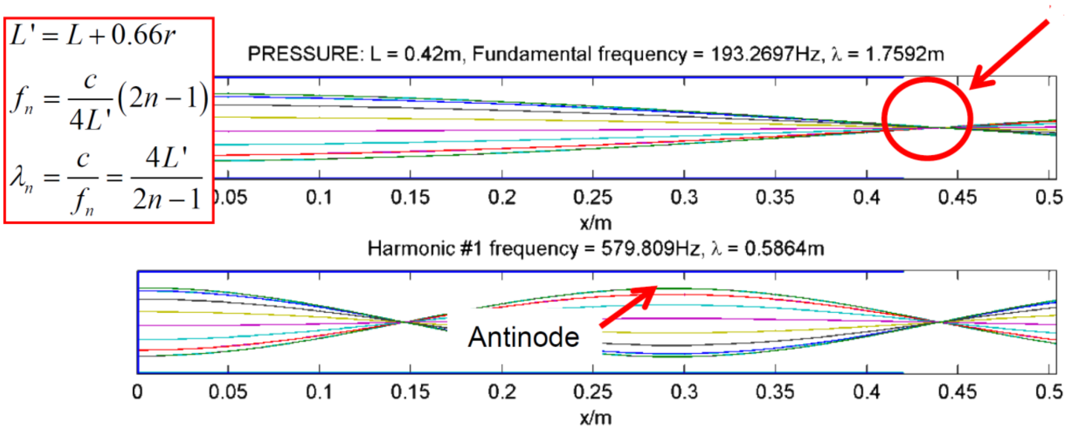
i.e. nodes and antinodes remain in the *same* spatial positions, but the wave amplitude varies in an oscillatory fashion with period T.

An **open-ended tube** such as a driven organ pipe must have an antinode of pressure at the driven end (i.e. a maximum variation above and below atmospheric pressure), and a node of pressure at the open end (i.e. at atmospheric pressure).

This means that the length of the pipe is an odd number of quarter wavelengths.

Note for pressure waves, an antinode of pressure corresponds to a node of particle displacement, and vice versa.

Open ended tubes also have an end correction due to the vibrations of the end of a tube (which cause slightly more air to vibrate). For tubes of radius r, the end correction means the effective length of the tube (i.e. from a standing wave perspective) is increased by about $\boxed{\delta L \approx \frac{2}{3}r}$.



In general, propagating waves will *lose energy* and be ***attenuated*** in amplitude (but not in frequency - we treat these 'Fourier components' separately). However, because energy may constantly be fed into a standing wave, it may achieve a '*resonant*' effect. This is the basis of sound production from most tuned instruments. In general, the **power** in a wave is given by: $\boxed{P = \frac{1}{2}ZA^2\omega^2}$ where Z is the '*wave impedence*.' i.e. double the amplitude and you quadruple the wave power.

Speeds of waves

Waves on a tensioned string: $c = \sqrt{T/\mu}$ where T is string tension and μ is the mass per unit length.

Elastic waves: $c = \sqrt{K/\rho}$ where K is the coefficient of stiffness (the 'elastic modulus') and ρ is the density.

Pressure waves in ideal gases: $c = \sqrt{\gamma p/\rho}$ where p is pressure, ρ is density and $\gamma = \frac{c_p}{c_v}$ is the ratio of specific heat capacities at constant pressure and volume. γ depends on the number of degrees of freedom of gas molecule motion, and can vary between 1.3 and about 1.7 depending on the molecule, and the temperature.

Sound waves in air travel about 340m/s. Sound waves in water travel about 1500m/s, and sound waves in rocks can travel at speeds between 5000m/s and 10,000m/s. The speed of light in a vacuum is $c = 2.998 \times 10^8\,\text{m/s}$.

Water waves

The *phase velocity* is as above: $c = \omega/k$. Because water waves of different wavelength can travel at different phase velocities, we also have the idea of a *group velocity* of a 'wave packet' $\boxed{c_g = d\omega/dk}$.

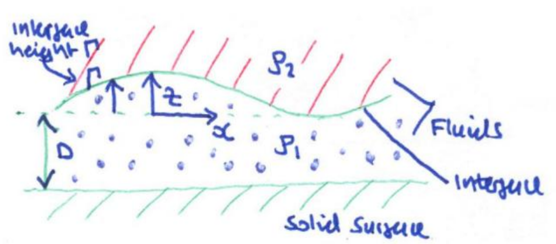
Deep water waves have *dispersion relationship* $\boxed{\omega^2 = gk}$, where g is the strength of gravity.

Shallow water waves have an alternative expression $\boxed{\omega^2 = gk^2 D}$ where D is the water depth.

Ripples (where *surface tension* σ cannot be ignored) have dispersion relationship $\boxed{\omega^2 = \left(\frac{\sigma k^3}{\rho} + gk\right)\tanh(kD)}$.

Note $\tanh(kD) \approx 1$ when $D \gg \lambda$ i.e. deep water waves. All these fluid waves assume that the density ρ of water is much greater than the density of the upper fluid (air). If this is not the case, the general dispersion relationship[4] is:

$$\boxed{\omega^2 = \frac{\sigma k^3 + g(\rho_1 - \rho_2)k}{\rho_2 + \rho_1 \operatorname{cotanh}(kD)}}$$



Question1

(i) Use 'freeze-frame diagrams' of the motion of a coiled-wire '*slinky*' to compare *longitudinal* and *transverse* waves. Indicate in both cases what the disturbance $\psi(x,t)$ corresponds to.

(ii) A guitarist plucks the bottom E string, which vibrates with fundamental frequency of 82.41Hz. If the speed of sound in air is 340m/s, calculate: (a) the period; (b) the wavelength; (c) the wavenumber; (d) the angular frequency of the sound waves in air produced by the vibrating string.

(iii) A wave disturbance is given by $\psi(x,t) = 3\cos(2.2x - 55t)$. Determine: (a) the amplitude; (b) the wavelength; (c) the frequency; (d) the period. Sketch the wave at time $t = 0$, and a small fraction of the period later.

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[4] $\operatorname{cotanh}(x) = 1/\tanh x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

(iv) Compare the wavelength of 50kHz ultrasound in water to 10GHz microwaves in air.

(v) A radio transmitter broadcasts Radio 2 at 88.1MHz and Radio 4 at 104.9MHz. If Radio 4 is broadcast at twice the power as Radio 2, what is the ratio of their amplitudes?

(vi) The speed of sound in air is measured at 340m/s. If $\gamma = 1.40$ and air pressure is 101,325Pa, calculate the density of air in g/cm^3.

(vii) The sound wave from a clap of thunder passes through a house to a physics student working in the basement. The sound has to pass through 2.0m of wood ($K =$ 28.3GPa, $\rho = 700\text{kgm}^{-3}$), 3.0m of glass ($K =$ 105GPa, $\rho = 2500\text{kgm}^{-3}$), and 1.5m of concrete ($K =$ 58GPa, $\rho = 2400\text{kgm}^{-3}$). How long does it take for the thunder to propagate through the house?

(viii) The A string of a conventionally tuned guitar vibrates at fundamental frequency 110.0Hz. If the string density is 6,610kg/m^3, string diameter = 0.036 inches (1 inch = 0.0254m) and string tension is 94.3N, calculate the length of the string between the nodes set by the *nut* and *bridge* of the guitar. In order to play better sounding 1980s rock guitar, a guitarist de-tunes all strings by one semitone. Assuming this means dividing all frequencies by $\sqrt[12]{2}$, calculate the new string tension of the string which now has a fundamental frequency of Ab.

(ix) Calculate the frequency (in Hz) of the fundamental and the first three harmonics of a *didgeridoo* that is 1.2m long, with a 8cm diameter open end. Assume the speed of sound in air is 340m/s.

(x) Show that the phase velocity of shallow water waves, (where surface tension can be ignored) is $c_p = \sqrt{gD}$. A wave of amplitude 2.0m is travelling towards the shore at a depth of 20.0m. It hits a reef which reduces the depth to 3.0m. What is the wave speed over the reef? Assuming no loss of wave power, calculate the amplitude of the wave over the reef. (*Hint*: assume the amount of water moving per second remains the same).

Question 2 An Alpine horn, or *labrophone*, of length $L = 2.4\text{m}$ is essentially a pipe with one end open to the atmosphere. Ignore any end-correction.

(a) Using appropriate diagrams involving standing pressure waves in an open-ended tube, driven at one end, explain why standing waves of wavelength λ_n must obey the equation $L = (2n-1)\frac{1}{4}\lambda_n$.

(b) Taking the speed of sound (dry air at 20^{o}C) to be $c = 342\text{ms}^{-1}$, determine the first nine 'resonant' frequencies /Hz. Determine the equi-tempered musical note for the last three of these (well, the closest) using the formula: $f_m = 440\text{Hz} \times 2^{\frac{1}{12}m}$.

Note	A	A#/Bb	B	C	C#Db	D	D#/Eb	E	E#/Fb	F	F#/Gb	G	G#/Ab
m	0	1	2	3	4	5	6	7	8	9	10	11	12

Note G#/Ab is an *octave* above the Ab, one *semitone* below the 440Hz A.

(c) If the $n = 7$ harmonic actually corresponded to exactly 440Hz, what would the horn length L have to be extended to?

Question 3 Use the trigonometric identity $\sin(A \pm B) = \sin A \cos B \mp \cos A \sin B$ to prove the standing wave result:

$$\psi(x,t) = \underbrace{A\sin(kx - \omega t)}_{\text{incoming wave}} - \underbrace{A\sin(-kx - \omega t)}_{\text{reflected wave}} = 2A\sin kx \cos \omega t .$$

Question 4

A guitar string of density ρ , length L and diameter d vibrates at fundamental frequency f . Show that the string must be under tension $T = \pi\rho(fLd)^2$. A *Fender Precision Bass* has $L = 0.864\text{m}$, and the thickest string (of diameter 2.67mm) is typically tuned to E (41.204Hz). Assuming it is made from steel of density 7850kg/m^3, calculate the string tension (in N). Assuming all the bass strings are under the same tension, and have the same length and density, calculate the diameter of the other three strings if their fundamental frequencies are: A (55.000Hz), D(73.416Hz) and G (97.999Hz).

Question 5

The 64 foot 9 inches (19.74m) *Diaphone-Dulzian* pipe[5] in the Boardwalk Hall Auditorium organ in Atlantic City, New Jersey, is one of the longest 'stops' in the world. It can produce a very low notes that are apparently "more felt than heard". The open end has an area of 1296 square inches. (1inch = 2.54cm). If the speed of sound in air is 340m/s, determine the fundamental and the first four harmonics of this pipe. Assume the open end is circular for the purposes of calculating the end correction.

Question 6

A set of tuning forks are held just above a vertically mounted plastic tube whose lower end is immersed in water. The plastic tube can be shortened or lengthened using a retort stand, boss and clamp, until a resonance is heard by placing the ear right next to the top of the tube.

(a) Draw a labeled diagram of the setup.

(b) Show that the first resonance is heard when the distance of the end of the plastic tube above the water is $L_1 = \frac{c}{4f} - \frac{2}{3}r$ where f is the tuning fork frequency, c is the speed of sound in air and r is the radius of the pipe.

(c) Using the following data, plot an appropriate straight line graph that, ideally, should pass through the origin and have a gradient being the speed of sound. $r = 9.9\text{mm}$.

$$f = [256, 271.2, 288, 304.4, 320, 341.3, 362, 384, 406.4, 426.6, 456.1, 480, 512]$$
$$L_1 = [325, 313, 292, 274, 269, 245, 235, 225, 208, 199, 190, 177, 165]$$

The frequencies are in Hz and the lengths in mm.

Question 7

(a) For deep water waves, show that the *group velocity* is exactly half the phase velocity.

(b) For deep water *ripples*, show that the phase velocity is: $c_p = \sqrt{\left(\frac{2\pi\sigma}{\lambda\rho} + \frac{g\lambda}{2\pi}\right)}$ and this means that these ripples have a minimum phase velocity of: $\sqrt[4]{\frac{4g\sigma}{\rho}}$.

(c) Use a spreadsheet to accurately plot c_p vs λ, using $\rho = 1000\text{kgm}^{-3}, g = 9.81\text{Nkg}^{-1}$ and $\sigma = 0.0728\text{Nm}^{-1}$. i.e. the parameters for water. Calculate the minimum phase velocity in cm/s. Overlay an additional trace for ripples on the surface of liquid mercury ($\rho = 13{,}600\text{kgm}^{-3}$, $\sigma = 0.487\text{Nm}^{-1}$). Use graph axis scales of $0.0 \le \lambda/\text{cm} \le 5.0$ and $0 \le c_p/\text{cms}^{-1} \le 50$.

Question 8

Show that the following wave representations all obey the *wave equation*: $\frac{\partial^2\psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$

(a) $\psi(x,t) = A\cos(kx - \omega t - \phi)$

(b) $\psi(x,t) = 2A\sin kx \cos \omega t$

(c) $\psi(x,t) = Ae^{i(kx - \omega t - \phi)}$

[5] http://www.oddmusic.com/gallery/om02700.html