Work done W is the *area* under a force f vs displacement x graph, assuming force and displacement vectors are *parallel* to each other. In calculus notation: $W = \int_{x_a}^{x_b} f(x)dx$. If the vectors $\mathbf{f}$, $\mathbf{x}$ are *not parallel*, then $W = \int_A^B \mathbf{f} \cdot d\mathbf{x}$ where $d\mathbf{x}$ is a vector displacement along the path between locations A and B.

If the force is a *constant*, then this reduces to the simpler idea:

"work done is force $\times$ distance moved in the direction of the force."

Power P is the *rate at which work is done*, so $W = \int_{t_A}^{t_B} P dt$. Now $W = \int_A^B \mathbf{f} \cdot d\mathbf{x} = \int_A^B \mathbf{f} \cdot \frac{d\mathbf{x}}{dt} dt = \int_A^B \mathbf{f} \cdot \mathbf{v} dt$, so *instantaneous power* $P = \mathbf{f} \cdot \mathbf{v}$. If force is parallel to velocity: **"Power = force $\times$ velocity".**

The relationship between *power* and *driving force* can be very useful in analyzing the use of *engines* to drive machines.

Example: If a vehicle of mass m is travelling at speed v up a slope of angle θ and subject to drag force $F_D = \frac{1}{2} c_D \rho A v^2$, let the driving force be $D = \alpha P / v$ where α is the efficiency of the engine, which converts the chemical potential energy of fuel at rate P watts. A is the cross sectional area of the vehicle facing the air, c_D is the drag coefficient and ρ is the density of air.

Hence by Newton II : $0 = D - mg\sin\theta - \frac{1}{2} c_D \rho A v^2 \quad \therefore P = \frac{1}{\alpha}\left(mgv\sin\theta + \frac{1}{2} c_D \rho A v^3 \right)$

Efficiency η is the *ratio of useful work done to total input energy*. This could also be expressed as the ratio of powers, and therefore be a quantity that could vary with time. Example, the rate of change of gravitational potential energy (i.e. work done per second against gravity) of a climber is 120W. The rate at which energy is converted from glucose in the body is 300W. Hence the efficiency is $\eta = 120/300 = 0.4 = 40\%$.

Unless otherwise stated, assume the strength of gravity is $g = 9.81 \text{Nkg}^{-1}$.

Reminder about **sliding friction:** If an object slides along a surface with coefficient of friction μ, the friction force is $F = \mu R$ where R is the normal contact force with the surface.

Question 1

(i) In building Stonehenge, a 25 (metric) tonne stone is dragged along rough flat ground using ropes in inclined at 15° to the horizontal. The stone is dragged at 1.5m/s and the coefficient of sliding friction is $\mu = 0.6$. Calculate the work done (in J) per second in pulling the stone. (*Hint*: Draw a force diagram. Then find the rope tension by eliminating the normal contact force between Newton II expressions in horizontal and vertical directions).

(ii) The Llanberis Path to the Summit of Snowdon, the highest mountain in Wales, has a height gain of 965m and has an average gradient of 1 in 7.1. The running record is about 40 minutes. Assuming a runner has a mass of 75kg, calculate the average rate of work done against gravity (in W), and hence the driving force (in N) parallel to the slope if resistive forces can be ignored.

(iii) The Snowdon *rack & pinion* railway, which follows the Llanberis path, has an average gradient of 1 : 7.86 and is 7.53km long. The journey takes 45minutes and the single carriage is pushed by small steam engine with power of 238kW[1]. Assuming the power quoted equates to the rate of work done against gravity, calculate the maximum mass of the engine + carriage + passengers (in kg).

(iv) A cyclist applies a constant torque about the turning axis of the back wheel. The cyclist applies a force of 200N on alternate pedals, at a radius of 0.17m from the wheel axis. If the mechanical power transformed is 180W, calculate the number of rotations of the pedals per minute.

[1] https://snowdonrailway.co.uk/abt-rack-railway-info/

(v) A ten-tonne truck travels up a hill of elevation angle $\theta = 3^{\circ}$ from the horizontal at a *constant speed* of 50km/h. The drag force it experiences is 50% of the component of weight acting down the slope. Work out an expression for the driving force acting on the truck and hence calculate the engine power (in kW), assuming it is 25% efficient.

(vi) In a Micro-Hydro Village Electrification (MHVE) project in Nepal, a water wheel is placed in the stream of a fast flowing river. The wheel drives a turbine which produces an output power of 27kW. If the density of water is 1000kg/m^3, and the wheel water inlet has a cross-sectional area of 0.5m^2, calculate the speed of the water flowing into the wheel. Assume the system is 20% efficient at converting the kinetic energy of water into electrical energy. Also calculate the vertical height the water must drop to attain this speed, ignoring losses.

(vii) Greenbeard the Pirate accountant orders a large bag of gold to be dragged from a deep pit dug into *Treasure Island.* The bag weighs 1234kg and the slope to the pit is inclined 30° to the horizontal. If the coefficient of friction between the bag and the slope is 0.3, and Greenbeard orders the bag to be pulled up at 1.2m/s, calculate the mechanical power required of the winch motor.

(viii) Jill watches in horror as Jack falls down the hill, spilling all the water and twisting his ankle. Jill gathers her pail of water, sits on a heavy rug, and proceeds to slide down the hill. Jill starts from rest, and at the bottom of the hill she is moving at 1.5m/s. She spills nothing, and has slid 20m.. Calculate the coefficient of sliding friction μ if the slope of the hill is 10° inclined from the horizontal. What is μ if Jill slides at 3.0m/s at the bottom of the slope?

(ix) A jet aircraft contains a pair of engines rated at 90kN each. (a) Calculate the total power demanded of *each* engine (in kW) if the aircraft flies directly upwards at 250m/s, which is as fast as it can go. (b) .If the aircraft has an effective wind-facing cross sectional area $A = 2.0\text{m}^2$, drag coefficient $c_D = 0.05$, calculate the mass of the aircraft. Assume the density of air is $\rho = 1.275\text{kgm}^{-3}$ (i.e. at ground level. Note the air density will reduce significantly has the aircraft climbs).

(x) A spacecraft of mass 12,000kg has a thruster which changes the kinetic energy of the rocket by 2.0MJ every second. Calculate the acceleration of the rocket (in m/s^2) when it is moving through empty space at 10km/s. *Note this is not really how rockets work! To achieve thrust, mass is ejected from a rocket. Conservation of momentum implies there is a thrust upon the rocket equal (and opposite) to the rate of momentum change of the ejected mass.*

Question 2 An electric car has a mass of 2,200kg. It has a battery consisting of thousands of lithium-ion cells which together contain a maximum energy of 360MJ. The electric motor can supply a maximum power of 615kW. The width of the car is 2.00m and the height is 1.44m. Assume the drag coefficient is $c_D = 0.1$, and the density of air is $\rho = 1.28\text{kgm}^{-3}$.

(a) If the car travels horizontally at a fixed speed of 70mph (31.3m/s), calculate the maximum range that the car can travel on one charge. Assume the motor converts electrical energy into mechanical power with efficiency of 26%

(b) Show the top speed of the car (in mph), based upon the maximum engine power, is about 213mph.

(c) If the top speed is reduced by 10%, what is the gradient (in angle of incline) of a straight slope?

Question 3 A solar panel subject to the full glare of the Sun (near the Earth) may absorb about 1.4kW of power per square meter. If this is converted into electrical energy with efficiency 50% and used to power some form of ion thruster, calculate the radius (in m) of a circular solar panel needed to accelerate a 100,000kg spacecraft from rest to 20km/s in 1 hour, assuming the solar panel is included in the mass, which remains essentially fixed during the acceleration. Ignore any effects of gravity or drag on the spacecraft. Also assume the 1.4kW flux of solar radiation remains constant.

Question 4[2] The drag force on an aircraft travelling at speed v is: kv^2 at ground level, $0.55kv^2$ at 6,000m and $0.30kv^2$ at 12,000m.

(i) If the aircraft, flying horizontally, can cruise at 220m/s at 12,000m, what speed (in m/s) will it travel at 6,000m if the engine power is unchanged?

(ii) Let $k = 2.5$ and the aircraft mass $m = 250 \times 10^3\text{kg}$. The takeoff speed is $v = 80\text{ms}^{-1}$ and the aircraft climbs at angle θ from the horizontal. At takeoff the engines develop *three times* the cruising power. What is θ (in degrees)? Hence calculate the lift force in kN.

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[2] Question 4 is adapted from Quadling *Mechanics 2* pp34