Schools Direct Assignment 1
Intervention for Learning: Literature and Practice
Teaching Reasoning and Proof

Dr Andrew French

January 3, 2014

1 Summary

This essay is in two main parts and focusses on the teaching of mathematical reasoning and proof in secondary schools. In the first part I review current literature on the subject via the following questions:

1. What is mathematics and mathematical reasoning?
2. What is mathematical proof? What are its wider roles in teaching?
3. How is mathematical proof taught in schools?
4. What are teachers' conceptions of mathematical proof?
5. What are students' conceptions of mathematical proof?

In the second part I shall reflect on my personal experiences relating to the teaching of mathematical proof, and link these to what I have surveyed in the literature.

2 A review of literature relating to the teaching of mathematical reasoning and proof

2.1 Mathematics and mathematical reasoning

2.1.1 "Intricately related structures"

Jones (2000) describes Mathematics as a "field of intricately related structures". These *structures* (or what Tall (1994) might describe as 'objects') come in various forms: Numbers, vectors, matrices, tensors, graphs, points, lines, curves, polygons, surfaces, solids, fields, are but a selection of the mathematical smörgåsbord. Thurston (1994) speaks of "*formal* patterns," which are at the heart of both definitions and interrelationships between mathematical objects.

A mathematical structure such a circle is defined *precisely* and *unambiguously* in terms of *already accepted objects*. "A closed curve which is the locus of a point, confined to a plane, which is a fixed distance from a fixed origin" is possibly sufficient, but of course builds upon definitions of the terms *curve, point, origin, locus, plane* and *distance*. Ultimately, our hierarchy of definitions will reduce to a series of fundamental axiomatic "polite fictions" (Thurston, 1994) which are agreed by all to be 'self evident.' A *point* might well be one of these, but *distance* is not, since it depends on the metric (i.e. curvature) of the space in which the points exist. For example, we could draw a circle on the surface of a balloon, but restrict our geometry to two-dimensional movements on the surface. In this case, the canon of geometrical relationships proven by Euclid in the *Elements* will not be generally applicable, as Euclid's 'fifth' postulate on parallel lines is no longer true for our balloon surface.

2.1.2 Theorems and proof: an edifice of truth

The interrelationships between mathematical objects are based upon *theorems*, edifices of truth built upon accepted facts. This is perhaps the defining and most useful feature of *mathematical reasoning* (i.e. the methodology of doing mathematics); that nearly every mathematical statement can be *proven* to be correct (or not) via a mechanistic chain of logic which connects the result to fundamental axioms. The 'nearly' prefix corresponds to the unfortunate reality (certainly for the twentieth century *Formalists* of Hilbert, Russell and Bourbaki *et al*) of Gödel's Incompleteness Theorem(s), which state that there will "always be theorems about the natural numbers that are true, but that are unprovable." (Paraphrased from *Stanford Encyclopedia of Philosophy*, accessed 19/12/13). In other words, the set of truths of mathematics is actually bigger than those which one can prove from any set of fixed axioms. However, the set of theorems we *can* prove has been shown to be sufficiently vast as to be dazzlingly useful to describe the physical world as the fundamental language of Science. Although *no scientific theory is provable*, merely inferred to be *likely* given a weight of repeated experimental evidence, nonetheless the associated syntactic transformations of mathematical symbols which correspond to physical quantities *can be proven to be true* since they abide by the rules of mathematics (Bogomolny 2013). For example, the parabolic trajectory of a particle launched in a uniform gravitational field in the absence of a resistive force *can be deduced* from Newton's Second Law of motion. The scope of its applicability is to be assessed via experiment, but where discrepancies are found between model and measurable reality (e.g. it is windy and the particle is a feather) it is always the fundamental assumptions which are modified (i.e. more forces accounted for in Newton's Second Law) rather than questioning the validity of the mathematical process of symbol manipulation.

$$m\frac{d^2}{dt^2}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -mg \end{pmatrix} \tag{1}$$

$$\left.\frac{dx}{dt}\right|_{t=0} = v\cos\theta \tag{2}$$

$$\left.\frac{dy}{dt}\right|_{t=0} = v\sin\theta \tag{3}$$

$$\Rightarrow y = x\tan\theta - \frac{g}{2v^2}\left(1+\tan^2\theta\right)x^2 \tag{4}$$

The parabolic equation of projectile motion (4) can be *deduced* from Newton's Second Law (1) written as a second order differential equation in terms of position vector $\begin{pmatrix} x \\ y \end{pmatrix}$, given initial conditions (2) and (3) which involve the the initial projectile speed v and elevation θ above the horizontal x axis. Note it is also assumed that the projectile has position vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ when time $t = 0$.

2.2 What is mathematical proof?

2.2.1 Mathematical proof: a formal verification mechanism or social consensus?

The *Oxford English Dictionary* (2005) defines *Proof*, in a mathematical sense, as "evidence or argument establishing a fact or the truth of a statement" or "a series of stages in the resolution of a mathematical problem." In the formal sense, proof is the means of *verifying* the truth of a theorem (Cadwallader-Olsker 2011, de-Villiers 1999). It is a scheme for deducing the truth of a mathematical proposition from other statements which are known to be true. (Bogomolny, 2013). Although the syntax of mathematics means a mechanistic derivation from fundamental axioms of, for example, *Pythagoras' Theorem* is possible (albeit rather unwieldy, at eighty pages!), in practice a proof is an argument that is "accepted by other mathematicians." (Cadwallader-Olsker, 2011). Herch (1993) augments this view with a definition that proof offers "an argument that convinces qualified judges." In other words, his definition is similar to the notion of proof in a legalistic or crime-and-punishment context. Of course, what "qualified" means is left ambiguous. However, 'cleared for publication in a respected peer-reviewed journal' is perhaps not a bad extension, since this mechanism has worked pretty well for humanity since the Royal Society was established in 1665. (Note Benos, (2007) cites *Medical Essays and Observations* published by the Royal Society of Edinburgh in 1731 as the first *printed* peer-review journal).

In proving a given theorem, there appears to be a degree of subjectivity with regard to what is an appropriate axiomatic point-of-departure. Cadwallader-Olsker (2011) describes mathematics as a "distinctly human endeavour," and that formal, syntactic proof is a unnecessary "shackle." In the case of Pythagoras' theorem, a large number of alternative, but equally true, proofs are possible. (Chambers, 1999; NRICH...). We are not restricted to the eighty page 'formal' version. Creative mathematicians and schoolchildren rejoice! I see a connection here with computer science. If computer programming could only proceed via manipulation of binary memory registers, it would be extremely time consuming to encode anything but the simplest of functions. The information technology creative explosion of the past forty years is fundamentally dependent on the creation of 'high level' languages (such as C#, Java, Python etc) which translate human understandable structures to machine code prior to being 'run' by a computer. It is perhaps no surprise that both the structural elements and syntax of most modern computer languages are highly mathematical in nature.

2.2.2 Wider role of proof in mathematics

To quote Lockhart (2002):

"Mathematics is the art of *explanation*. If you deny students the opportunity to engage in this activity– to pose their own problems, to make their own conjectures and discoveries, to be wrong, to be creatively frustrated, to have an inspiration, and to cobble together their own explanations and proofs– you deny them mathematics itself."

The view that proof has other functions, i.e. in addition to the verification of mathematical facts, appears to be widely held in all the literature I have surveyed. Alcock (2004) describes the importance of 'intuitive example objects' in proof construction, whereas de-Villiers (2012) proposes a broader suite of features that should be associated with the role of proof, which are described in Table 1 below. I have commented on each aspect.

Function of mathematical proof	Explanation
Explanation	Proof is constructed in such a way to maximize the clarity of the constituent mathematical arguments. *Why* a theorem is true is emphasised over brevity. Whereas a 'convincing' proof might be sufficiently encoded in a terse collection of symbols only familiar to a very small group of mathematicians (Hersh, 1993; Thurston, 1994) an 'explanatory' proof might make use of examples, diagrams and use descriptions in everyday language. (Hoyles & Healy, 2000).
Discovery	Hanna (2000) describes proof as "a way to display the mathematical machinery for solving problems." Changing aspects of this internal machinery can lead to new conjectures. According to Hoyles & Jones (1998) and Leikin (2013), dynamic "mouse transformable" geometry software can be a practical way of investigating this aspect in a classroom.
Verification	The traditional function of proof. *Conjectures* can only become accepted as *theorems* once they have been proven to be true. The latter, despite attempts by *Formalists* such as Hilbert and Bourbaki, can never be a fully automated process for both fundamental (i.e. Gödel's Incompleteness Theorem) and practical reasons of human comprehension. According to Hersh (1993), "an argument that convinces qualified judges" constitutes a proof.
Intellectual challenge	In *Beobachtungen über das Gefühl des Schönen und Erhabenen (Observations on the Feeling of the Beautiful and Sublime)* (1764) the philosopher Emmanuel Kant associates intellectual discovery with the 'finest' of sensory experiences. Understanding a proof, and ideally devising your own, approaches this ideal. Revealing the mysteries of the Universe has proven to be an addictive occupation for many people across the centuries! It has also yielded enormous utility to humanity, and immortality for those who are most strongly associated with important discoveries.
'Systematization'	The collection of mathematical ideas, and proofs, into a *system* can result in a highly interconnected schema, from which many theorems (and thus practical applications) can follow from the same essential ideas. Euclid's *Elements* is possibly the most famous historical example, and has been an essential component of the study of geometry for nearly two thousand years.
Aesthetics	Many proofs have a combination of brevity and clarity that are often described as aesthetic, i.e. *beautiful* in both function and form. The typographic and or diagrammatic representations of mathematics inherent in proofs are often considered as works of art in themselves. Conversely, much of art is mathematical either overtly (Mondrian, Escher) or in construction (anything that uses perspective!)
Potential for algorithm generation	The mechanism of a mathematical proof can often yield recipes for rapid computation via machines. For example, the overall concept of calculus can be approximated as finite differences and thus used to devise methods to solve almost *any* differential equation, including ones which are hard (or possibly impossible) to solve by analytical means. The implication for physical sciences and engineering is enormous, since nearly all physical laws can be stated as differential equations.
Memorisability	A proof is essentially a story. If the characters and their interrelationships are sufficiently well established, and the number of interactions are relatively small, the proof is likely to be memorisable and the associated methods are added to the working repertoire of the mathematician. As in fiction, pictures sadly tend to be associated with works for young children and are deemed unnecessary for adults.

Figure 1: Additional functions of mathematical proof, as proposed by de-Villiers (2012).

The key thesis of de-Villiers (2012), and also Hanna (2000), appears to be an increased emphasis of the explanatory and discovery functions of proof. According to them, a concentration on "*what* is mathematics" diminishes its power. I am inclined to agree. Investigating *why* the product of an odd and even integer is even reveals so much more than simply accepting it as a verifiable fact. The discovery function relates to the generation of new conjectures following modification of the axioms which constitute a proof. Stylianides (2006) states "proof is the principal means by which mathematicians establish truth and derive new knowledge from old". This is not confined to the intellectual zenith of research mathematics; Hoyles & Jones (1998) and Leikin (2013) describe use of dynamic geometry software in school classrooms as a practical tool for generating testable conjectures. (The efficacy of this approach shall be discussed in section 2.5 below).

Cadwallader-Olsker (2011) and Hanna (2000) take the view that "rigorous proof is secondary to understanding", and one should (certainly at secondary school level) select a proof for its explanatory value over other attributes such as an ability to convince a grand jury of mathematics professors. De-Villiers describes proof as "a means of communication, a forum for critical debate." This view is echoed by Ball *et al* (2002) who view proof as "an essential tool for promoting mathematical understanding." They stress the need to also link epistemological aspects, i.e. the process of how a proof is (and *was* in a historical context) constructed. Similarly, the Nuffield Foundation (accessed 18/12/13) poses a criticism of geometric reasoning being (merely) "deductive proving", since it "ignores the process by which new mathematics is created." In the opinion of the Foundation, the *essence of mathematical reasoning is associated with proof making* :

- Posing problems
- Solving problems
- Analysing examples
- Making and revising conjectures
- Searching for counter examples

Mathematics is perhaps *fractal* in the sense that a proof of a particular theorem, with the associated additional attributes mentioned by de-Villiers, is self-similar in philosophical terms to the entire subject. In contrast to what Hersh (1993) or Ball (2002) would describe as an "obligatory ritual without meaning", an *appropriately explored* proof can illuminate the beauty and power of mathematics in general, combined with the joy of personal discovery. Hersh commends the "humanist maths teacher" to "use the most enlightening proofs, not the most general or the shortest." This is indeed a ideal which I intend aspire to in my own teaching practice. A practical reflection upon this view is essentially the discussion in section 3.

2.3 How is mathematical proof taught in Schools?

Despite general agreement amongst mathematicians (and certainly researchers in mathematical education) that proof is a fundamental component of their subject, the prominence of proof in school curricula is often criticised as being "peripheral" (Knuth, 2002), leading to widespread misconceptions about the notion of proof (Harel & Sowder, 1998) and a lack of confidence and aptitude in University level mathematics students, since they encounter it too late in their development (Stylianides 2006, Jones, 2000). Jones describes a fairly bleak school reality where "proving is an obscure ritual," which, "disappears into a series of innocuous classroom tasks in which students 'spot patterns' but not much else." Hoyles & Healy (2000) comment that the 'proofs' students *would adopt for themselves* are typically dominated by *empirical arguments*. For example: "The formula $S_n = \frac{1}{2}n(n+1)$ we discovered for the sum of the first n integers seems to work for the first ten numbers, so therefore it must be true!"

Although there have been steps to incorporate mathematical reasoning and proof into the UK National Curriculum (QCA, 2004), the rise of empirical pseudo-proof (certainly in the UK) may well be an unintended consequence of much earlier educational initiatives to set investigations and discovery learning at the heart of the school experience. In the million-copy-sold book *How to Solve it,* Polya (1945, 1957) places high value on empirical, pattern spotting, methods to generate conjectures. In his view, mathematicians only bother to pursue a rigorous proof once they have developed a *belief* that a proposition is likely to be true. Unfortunately it seems many secondary school teachers over the years have misinterpreted Polya's *heuristics* as a truth-yielding mechanism, rather than a method for generating *inspiration* for more rigorous, deductive, thought. Maher (1996, 2005) appears to advocate 'minimal teacher intervention,' applying the teaching strategy where is it the responsibility of the child learner to 'discover' the key mathematical relationships germane to the current topic. Although there seems to be plenty of evidence that the student being studied is engaged with her work in 'proof development', I am somewhat alarmed that the paper refers rather mysteriously to a "proof by cases", "a legitimate form used by mathematicians." What this really means is not properly explained. Perhaps I am projecting my own scepticisms, but the whole enterprise does seem to give the impression of an entertaining diversion for children that might impress a visiting inspector, but has limited long term mathematical value.

Cadwallader-Olsker (2011), Hanna (2000), Nyaumwe (2007), Stylianides (2006), Ball (2002) and others describe variations in the teaching of proof in different countries. Whereas Holland and the UK have approached proof via practical investigative and often empirical means, in the United States, France, Zimbabwe and Japan a more formal approach is often adopted. Prior to the NCTM Standards of 1989, Hanna (2000) states that proof was "essentially absent" from the American syllabus. Although there are varying approaches, the extreme perhaps being the "Modified Moore Method," where theorems are only allowed to be used when they have been discussed and agreed to be true by the class (Cadwallader-Olsker, 2011), the typical reality is that the last bastion of proof is geometry. Unfortunately, 'proofs' often take the form of 'polished', 'two-column' largely *Elements*-inspired descriptions with "no link to how the proof was constructed". (Cadwallader-Olsker, 2011). Combined with the habit in the US that geometry is typically taught in isolation to algebra, it is perhaps unsurprising that Harel & Sowder (1998) and Stylianides (2006) report that University students find proof generation difficult.

In Quebec, Zack (1997) describes the use of 'Math logbooks' and relatively effective investigative learning at elementary school level. However, he comments that the 'why' aspects of mathematical discoveries (such as the origin of the formula $\frac{1}{6}n(n+1)(2n+1)$ for the sum of the squares of the first n natural numbers, which was empirically determined by a group of students) were often not explored.

2.4 The concept of mathematical proof held by teachers

"Teachers believe that introducing proof in secondary schools might make the learning of mathematics difficult, especially for girls." (Nyaumwe, 2007)

"Secondary school maths teachers have a relatively impoverished grasp of the nature of proof." (Jones, 2000).

"One might say that it is mathematically unethical not to maintain the distinctions between casual reasoning and proof." (Jaffe & Quinn, 1993)

The literature surveyed seems to raise concerns that "empirical pattern spotting" is being taught as mathematical proof in many educational contexts. There is no concrete evidence that this practice is endemic, certainly in the UK, but the anecdotal observations appear to at least justify the debate. Perhaps more worrying is the low prominence of proof in mathematics curricula worldwide. One may infer (but not deduce!) a link between a possible unwillingness to teach proof to schoolchildren if the teachers themselves found (and still find) this aspect of mathematics difficult. Tall (1994) and Thurston (1994) provide an alternative angle on this issue by alluding to "dysfunctional communications" within the professional mathematical community. Thurston claims that, outside the community associated with a very specialised field, it is often almost impossible to follow their thoughts and ideas given the scope of communication channels (formal papers in journals and conference presentations) and the method of incentivisation of most academics; i.e. "theorem credits" rather than professional objectives relating to the wider explanation and dissemination of their ideas. In other words, since academics are paid in proportion to the numbers of papers they write (or possibly the number of citations) rather than the efficacy of their teaching, one may entertain the possibility that the next generation of school mathematics teachers are not being as educated as well as they could be. Peressini (2004) discusses a conceptual framework for teacher education in the United States, based upon diverse case studies. In essence, ongoing early-career professional development is recommended to avoid the problem of subject knowledge gaps, in particular the 'proof-prejudice' described above.

If, as Nyaumwe puts it, we omit proof because we believe students will find it difficult, where does this end? Do we largely ignore algebra until students are well into their teens because it is perceived as complicated? Do we omit calculus from A-Level physics because we must allow the subject to be accessible to those who don't also study mathematics? Unfortunately, in the UK we do all of these things. (For example see `http://www.bbc.co.uk/news/education-17854008`). Stylianides (2006) suggests very young children develop sufficiently in a psychological sense to handle the concepts of deduction and logic inherent in some meaningful exercise in mathematical proof. Are we as teachers guilty of projecting our own mathematical anxieties onto our students, creating a fiction of difficulty and confusion which can only perpetuate an erosion of mathematical aptitude from one generation to the next? Rather than avoid, shouldn't we, as de-Villiers and Lockhart would put it, just try harder to *explain* mathematics when our students are initially stumped? Are we just too preoccupied with students passing all-too regular examinations, (Tall, 1994) and live in fear that our professional efficacy will be judged poorly if everybody doesn't immediately 'get it'?

2.5 The concept of mathematical proof held by students

Harel & Sowder (1998) propose three *Proof Schemes* as a model for the concept of mathematical proof as held by students.

1. **External conviction proof schemes**
   (a) *Ritualistic* e.g. "The area of a circle is pi r squared, because it is (and we chanted this fact aloud in class until we could remember it)."
   (b) *Authoritarian* e.g. "Because our teacher told us it was true".
   (c) *Symbolic* e.g. "Taking the square root removes the little two from the index of a number, so $\sqrt{a^2+b^2} = a+b$, right?"
2. **Empirical proof schemes**
   (a) *Inductive* e.g. "this formula seems to hold for all the terms I have checked so far, therefore it must be true."
   (b) *Perceptual* e.g. "this looks like it could be a right angle. Therefore it is."
3. **Analytical proof schemes**
   (a) *Transformational* e.g. "I dragged the vertex of the circumscribed arrowhead around and the ratio between angles $A\widehat{B}C$ and $A\widehat{O}C$ remained the same!"
   (b) *Axiomatic* e.g. "The sum of the internal angles of a triangle is 180^{o}. I can divide up a pentagon into exactly three non-overlapping triangles, so therefore the sum of the internal angles in a pentagon must be $3 \times 180^{o} = 540^{o}$"

Although it is not stated explicitly, one can infer from Harel & Sowder's analysis that External conviction proof schemes are possibly widespread. I have no quantitative evidence to back up this view, but I certainly have anecdotal experience from my own practice of teaching that many pupils endeavour to resort to memorisation of formulae and methods instead of building a conceptual framework based on true understanding. This year I observed in my Year 9 class a stark difference in comprehension of simple word problems, compared to pure algebra questions of equivalent sophistication. I conjecture that the weaker students where thinking via a Symbolic proof scheme. Since the word problem did not immediately associate with previously experienced symbolic manipulations, they often ignored it rather than endeavour to attach mathematical meaning to the words. Weber (2007) reports a similar assessment of University level mathematics students. Those judged as 'more successful' "reformulated a concept definition first, then connected the concept to prior learning." i.e. they spent time understanding the question before diving into a proof.

Perhaps dangerous for the ego is Inglis' (2009) findings that perception of teacher authority has an impact upon whether students believe a proof to be true. I recall at University the use of "proof by intimidation" as a light hearted response by exasperated lecturers to annoyingly persistent question askers. Perhaps we should be careful of such a glib remark when addressing younger students!

As discussed above, many authors (Jones (1997) being possibly the most negative) allude to the prevalence of Empirical proof schemes, and the lack of evidence for widespread Axiomatic thinking, which surely must be the implied end-goal following teacher intervention. Although Leikin (2013) reports evidence of Transformational thinking facilitated by the use of dynamic geometry software, Hoyles & Jones (1998) caution this approach as likely to engender "naive empiricism." Ball (2002) offers a wider comment: "Clearly avoiding the deduction of perception is only one (of the) pitfalls in geometrical reasoning". In other words, dynamic geometry can be really useful as a discovery and investigative tool, but real mathematical understanding cannot happen until you draw a diagram and connect quantities (such as angles, lengths of lines etc) using known algebraic relationships. In other words, blend algebra and geometry into a cohesive, mathematical whole.

Sadly, resorting to algebra is not what Healy & Hoyles (2000) discovered in a survey of high attaining 14 and 15 year olds. Proofs they would adopt for themselves were dominated by empirical argument, although many were apparently aware of the limitations of this view. Proofs which contain "everyday language", diagrams and examples, were preferred although, worryingly, students held the impression that "proofs that would gain approval from their teacher" were the ones which "contain complicated algebra." As postulated in the previous section, perhaps the inference is that the students feel emotionally (or are conditioned socially to feel) that 'algebra is complicated' rather than 'complicated algebra is required.' There is certainly a difference!

3 A reflective account of my own teaching experience of reasoning and proof and how this relates to the literature reviewed

3.1 Method of analysis

In this section I shall review various occasions in my own teaching practice where mathematical proof has been an overt aspect of the subject matter being discussed. I will endeavour to link my reflections to the literature described previously. In a similar fashion to the 'Maths Log' alluded to by Zack (1997), I have begun what will hopefully become a sustained habit of assembling a 'mathematical mezze' of interesting and typically proof-related mathematical encounters during each year. The examples with * correspond to the 'mezze' of last year's teaching. I used this resource successfully during the final week of term for all my classes last year. In addition to a reminder of some of the most interesting connections made by the students, I think they also appreciated that I was also actively participating in a voyage of discovery. I think it is vital for students to realise that mathematics learning is never a static entity, for both them and their teachers. On a personal level, the requirements of teaching over the past two years have catalyzed significantly higher levels of technical creativity. To help better store and disseminate this information, I have created a website `www.eclecticon.info`, which I encourage my students to visit. Hopefully this will help to offset the "dysfunctional communication" between professional mathematicians as described by Tall (1994) and Thurston (1994).

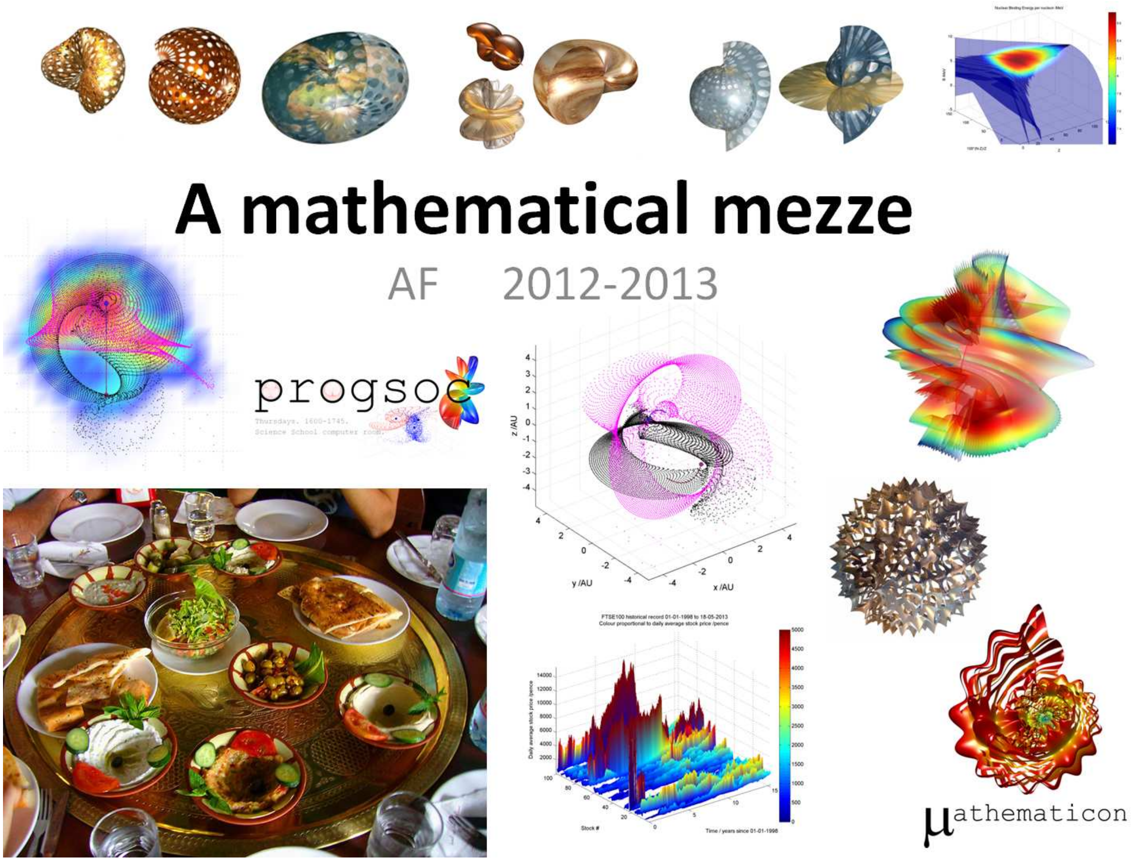


Figure 2: A summary of mathematical discoveries by myself and my classes during the teaching year 2012-2013.

Figure 3: Home page of my personal website `www.eclecticon.info`. "A veritable cornucopia of links, ideas and resources."

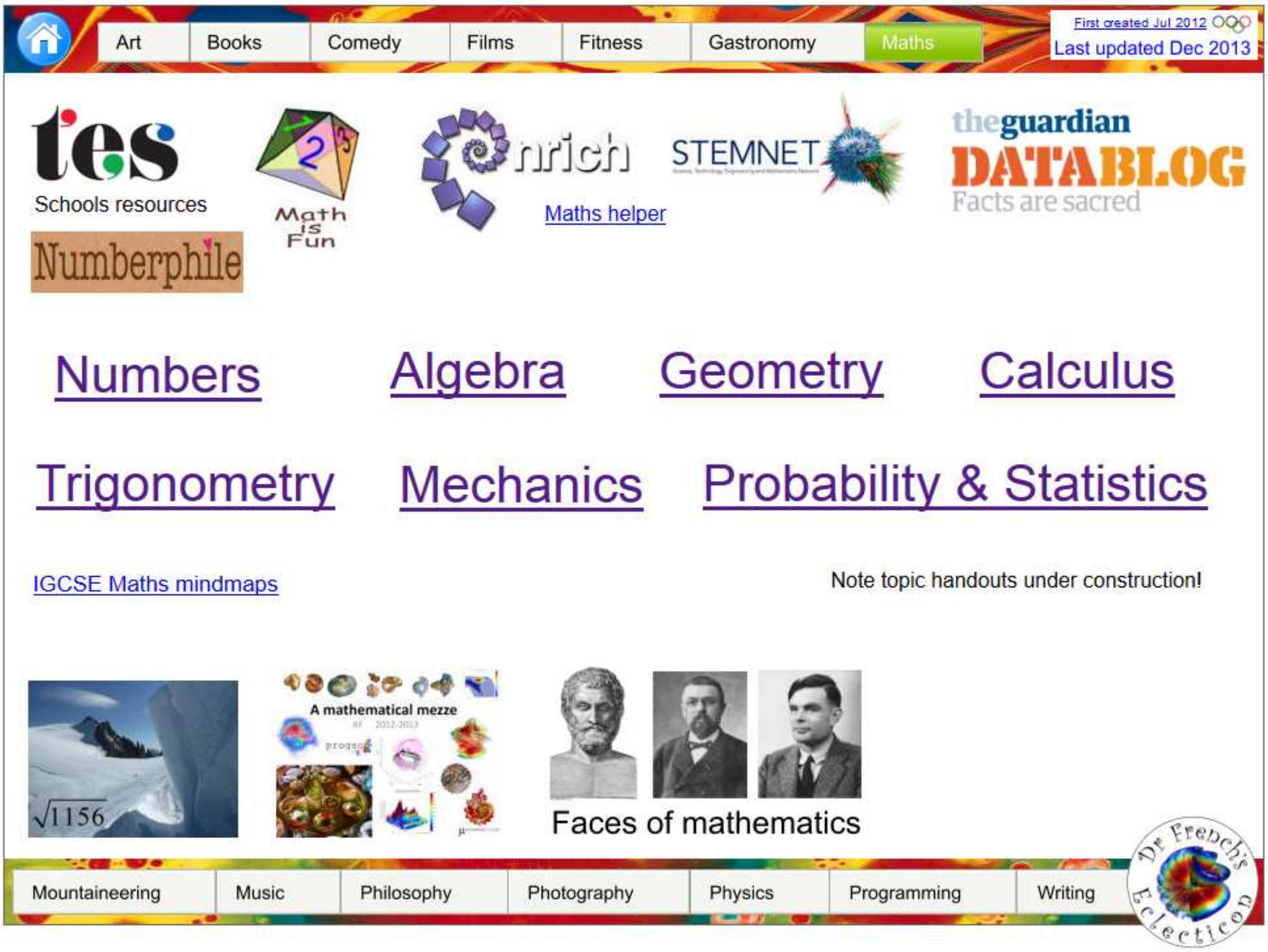


Figure 4: Maths topic page of my website `www.eclecticon.info`. This enables my students to download resource material and hopefully better appreciate the inter-connected structure of mathematics.

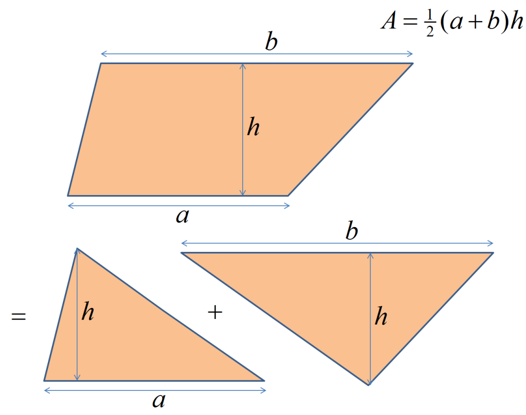


Figure 5: The area of a trapezium of width h and parallel sides of length a and b is $\frac{1}{2}(a+b)h$. This follows very elegantly from the area of a triangle ("one half base x perpendicular height") if one considers a single diagonal cut, thus representing the trapezium as the sum of two triangles.

3.2 *The area of a trapezium (Year 9)

(See Figure 5). This type of proof is what I tend to describe as ‘Egyptian mathematics’. In other words I express the sum of areas of a geometric figure in an algebraic sense. I do this for clarity, but it does seem to resonate with Tall’s (2006) notion of cognitive development. The visual constructions are of the ‘Conceptual Embodied’ world, whereas the algebraic representations build a bridge to the ‘Proceptual Symbolic’ world. As a class we firstly derived the area of a right angled triangle by dividing a rectangle. We then extended this, again by drawing and explaining, to a slanted triangle. By diagonal division of the trapezium, the final result was then obvious to all. They key step is the extension of the slanted triangle into a right angled triangle to enable, via some simple algebra, the area of the slanted triangle to be determined in terms of base and perpendicular height. This was *not* explored using dynamic geometry, rightly so since it is the idea of ‘how could we relate our more general triangle to something we already know about?’ which was the key question. The method of *adding to the construction*, rather than transforming it, was the strategy in this case.

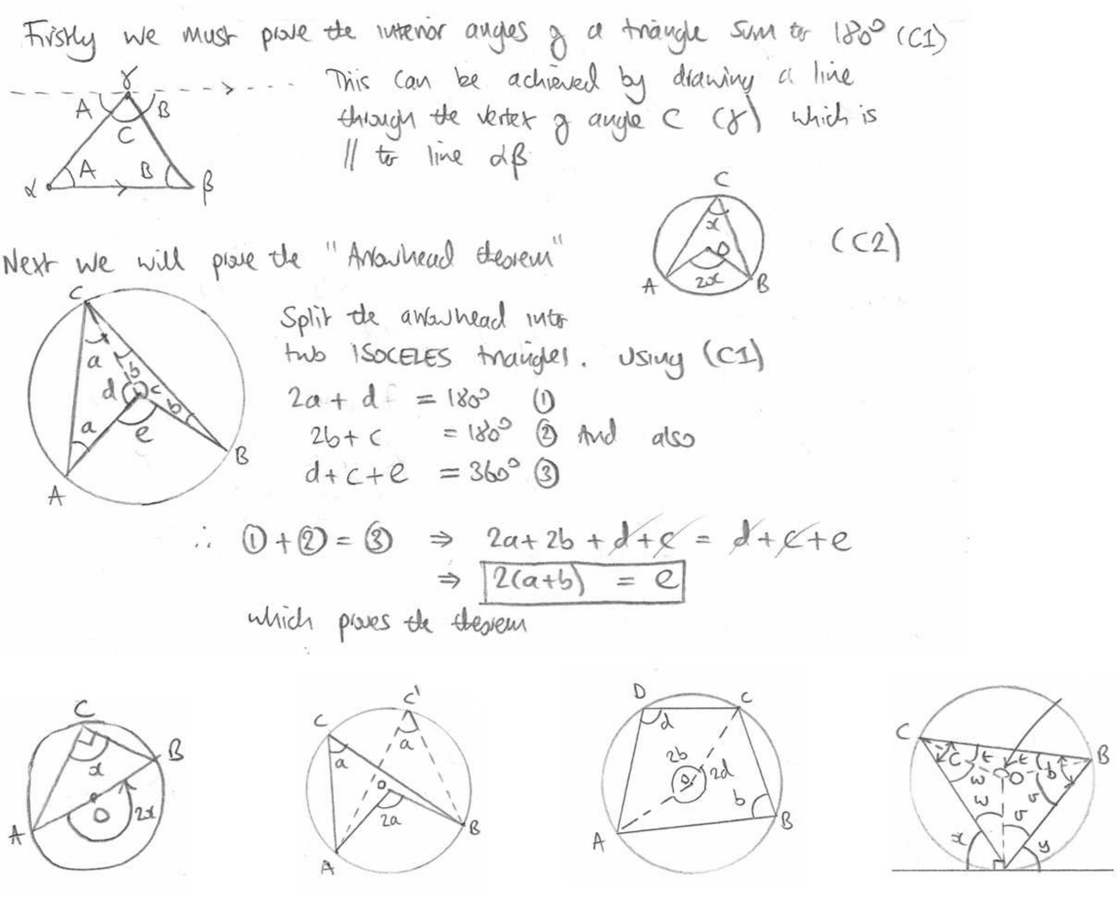


Figure 6: My variant on a Euclidean 'proof scheme' for circles theoems.

3.3 *Circle Theorems (Year 9)

(See Figure 6). For many topics I believe there is a strong case for "proof driven pedagogy", especially in what de-Villiers might refer to as 'systematised,' interconnected proofs. For circle theorems, there is a natural order, starting from the sum of interior angles of a triangle and culminating in the alternate segment theorem. In all cases I have endeavoured to (i) blend geometry and algebra as appropriate; (ii) use natural, descriptive language e.g. "arrowhead theorem" ; (iii) use pictorial referencing of previously proven theorems (i.e. I sketch the theorem rather than simply refer to by name). The differences between my version and something 'formal' akin to Euclid's *Elements* address the explanatory role of proof (de-Villiers *et al*) and the benefit of using natural language (Healy & Hoyles, 2000).

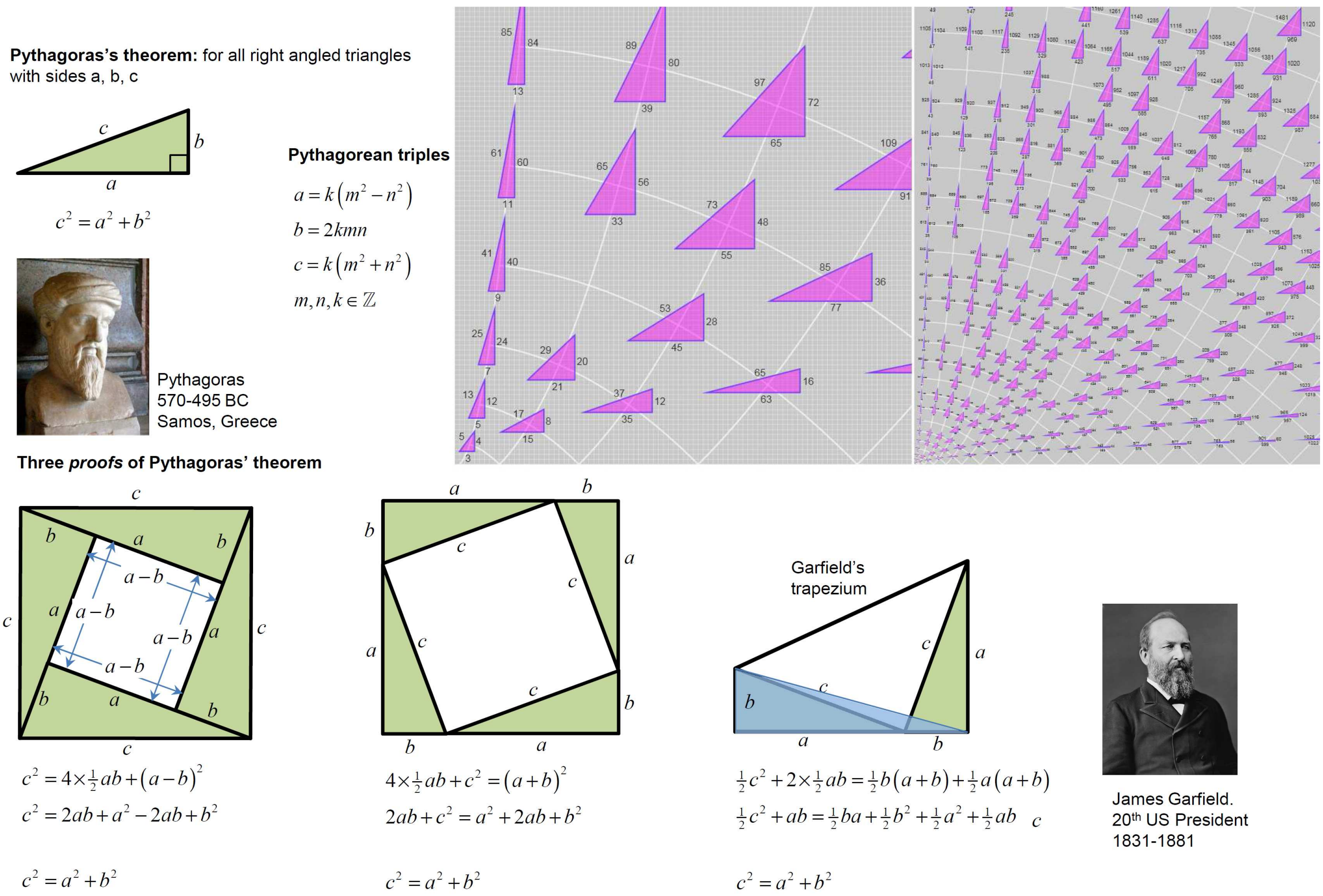
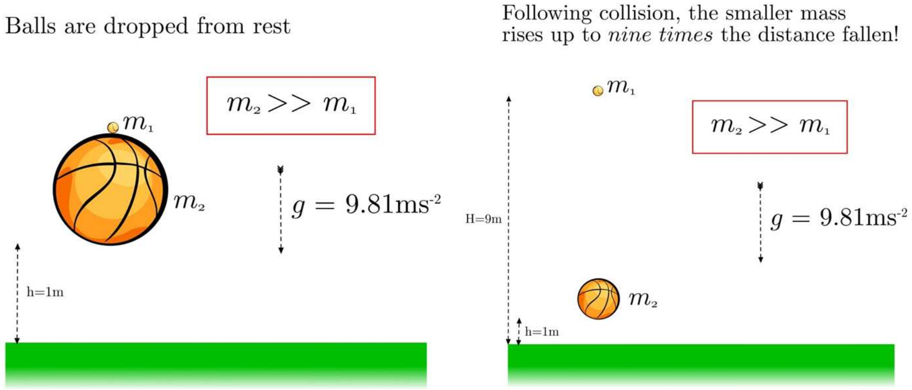


Figure 7: This figure is a snapshot of one of many 'topic handouts' I have prepared this year to condense key knowledge about each topic, and focus on the inter-connections. I strongly believe theorems should be able to be explained on their own, without complex referencing to other works. Each handout is therefore self contained, with only elementary results assumed.

3.4 Pythagoras's theorem and *Pythagorean triples (all years)

(See Figure 7). I have reviewed many approaches to the proof of Pythagoras' Theorem (Chambers, 1999, NRICH etc) and I have selected three versions which satisfy the criteria of being 'fairly obvious if you read the description.' I stress the importance of the blend of geometry and algebra, and the use of colour to highlight the geometric components of the constructions. Each of the three proofs involves computing the interior area of the boundary polygon in two different ways. The corespondance between each yields the Pythagorean result. The strategy here is perhaps as important as the theorem itself, as it provides students with a method which they could apply, heuristically, to a much wider variety of geometric problems.



$$v_1 = u\left(\frac{3 - m_1/m_2}{1 + m_1/m_2}\right) \qquad u^2 = 2gh$$

Figure 8: When a basket ball and tennis ball are dropped together onto a hard floor, the ensuing mostly elastic collision causes a dramatic velocity amplification of the tennis ball. In the limit of an infintely heavy lower ball relative to the upper ball, and a purely elastic collision, the upper ball will rise to nine times the original height. For the basket-ball and tennis-ball experiment, height ratios of between 3 and 4 times were observed.

3.5 *A interesting case of two body collisions and the 'Irish Moonshot'

(See Figure 8). This rather beautiful and surprising result was first demonstrated to my students *experimentally* using a basket ball and a tennis ball. In other words, motivation for a mathematical explanation is generated from an unexpected physical experience. An extension of this velocity multiplier phenomenon is to consider more than two balls. A playful analysis is to consider how many elastic balls are required (if each ball is twice the mass of the ball above) to cause the uppermost ball to escape the Earth's gravitational field. (The answer is around 26). The latter extension shows how the proof mechanism can stimulate discussion of a wider problem. The act of proving the mathematical result yields a deeper insight into the fundamental physics, and also yields computational tools when can then be deployed to solve more complex problems.

3.6 Combinatorics and the Binomial Theorem (Years 10 & 12)

We have recently begun a project at Winchester College to introduce ideas of simple combinatorics to Middle Part (Year 10). The goal is to expose students to the wonderful connections between (i) Pascal's triangle; (ii) Binomial expansions such as $(a+b)^4$ and (iii) problems like "how many ways can we arrange the letters of ABRACADABRA?", leading to "what is the probability of my Junior Colts football team winning more than half of their ten matches this year?"

The initial connections were made in a discovery sense, and certainly not proven for the year 10 students. Although I stated 'this fantastic result could or course be proven' I was certainly guilty of perpetuating an Authoritarian proof scheme (Harel & Sowder, 1998). However, as the course progressed, we, as a class, discussed that $\left(\begin{array}{c} n \\ r \end{array}\right) = \frac{n!}{(n-r)!r!}$ is the number of permutations of n letters of two types, where r are the number of one type and $n-r$ are the number of the other type. This *alternative description* of the number of combinations of r distinct objects taken from n enabled a very simple proof of the Binomial theorem to be presented to my Year 12 students:

Consider the expansion $(a+b)^n$ where n is an integer $n \geq 0$. From the algebraic meaning of the expression $(a+b)^n$, the expansion will be a sum of *all permutations* of $a^i b^j$ where i, j belong to the set of natural numbers $(\mathbb{N})$, 0 ..... n. Now $\left(\begin{array}{c} n \\ r \end{array}\right)$ tells us how many permutations of $a^i b^j$ are possible, where $i = r$ and $j = n - r$. Hence without further analysis we can write down the *Binomial Theorem* for $n \in \mathbb{N}$

$$(a+b)^n = \left(\begin{array}{c} n \\ 0 \end{array}\right) a^0 b^n + \left(\begin{array}{c} n \\ 1 \end{array}\right) a^1 b^{n-1} + ... \left(\begin{array}{c} n \\ n \end{array}\right) a^n b^0 = \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array}\right) a^r b^{n-r} \qquad (5)$$

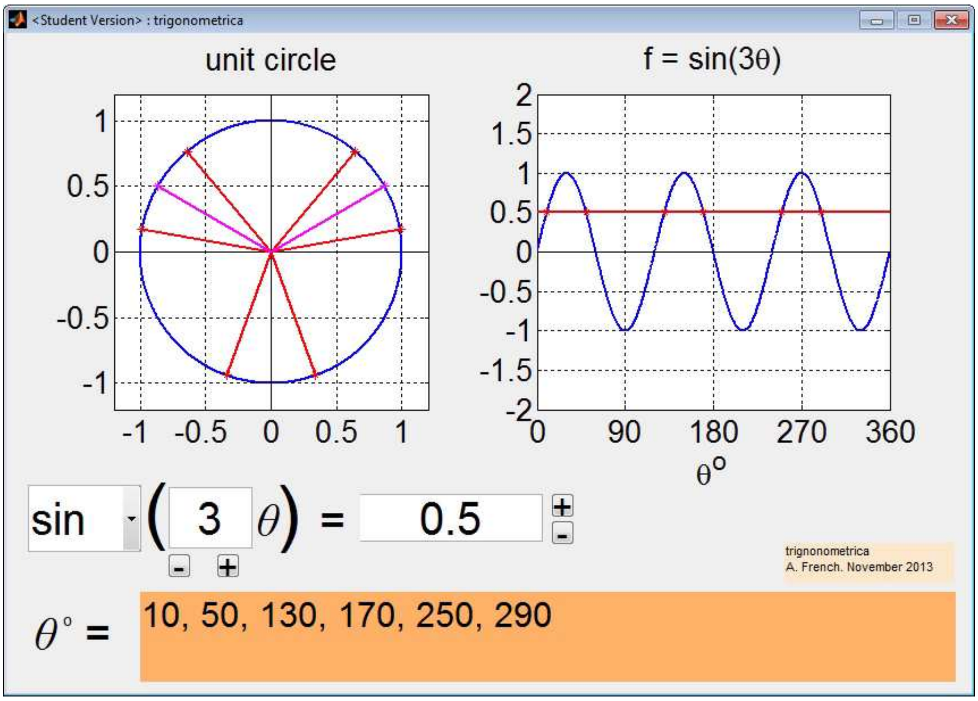
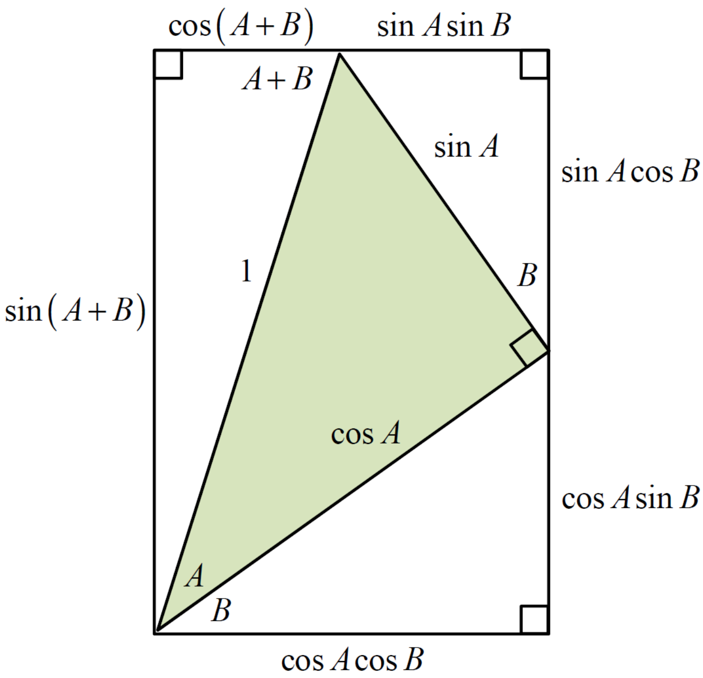


Figure 9: Dynamic software to enable the connections between the unit circle definitions of trigonometric functions and their 'wave' forms to be investigated in a classroom context.

3.7 Trigonometry - the unit circle (Years 10 & 12)

(See Figure 9). I have made more rigorous use of the unit circle as a definition of sine, cosine and tangent trigonometric functions in my teaching this year. In a similar way to the way Hoyles & Jones (1998) and Leikin (2103) allude to the use of dynamic geometry, I have created software which does the same for algebraic equations such as $\sin(n\theta) = k$. Animating the graph with changes in parameters n and k enable my students to build a more sophisticated 'cognitive framework' (see Tall, 2006), which I hope will yield greater understanding of how to solve these problems *without* a computer.



$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(\pm A) = \cos A$$
$$\sin(\pm A) = \sin A$$

odd and even relationship

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

Figure 10: A most elegant construction to enable the addition formulae for sine and cosine to be proven. Adapted from Wikipedia: `http://en.wikipedia.org/wiki/File:AngleAdditionDiagram.svg`

3.8 Trigonometry - addition formulae (Year 12)

(See Figure 10). I researched many different ways of proving the addition formulae for sine and cosine, as I have always found this topic to be somewhat messy. It has always been a 'do once' proof, certainly not one to be memorized by the students. I consider my new approach, blending geometry and algebra, quite beautiful in its simplicity. I will endeavour to set problems based upon it to facilitate retention (or 'memorisation') by my students. (See de-Villiers, 2012). Note a proof for the even and odd nature of sine and cosine manifested in a very natural way during a class discussion following how we would represent $\sin(\pm\theta)$ and $\cos(\pm\theta)$ using the unit circle.

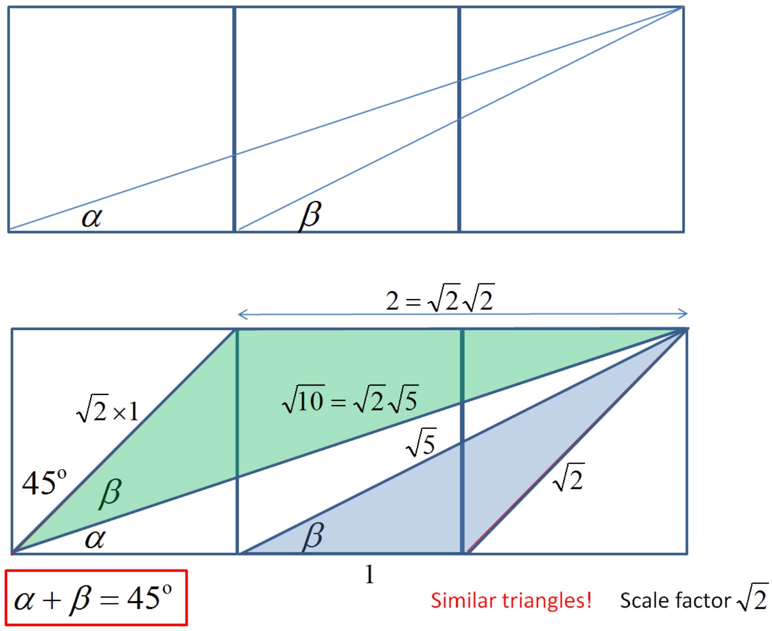


Figure 11: Irish Superbrain problem: Prove that the sum of angles $\alpha + \beta = 45^o$.

3.9 *Three squares 45° angle problem (Year 10)

(See Figure 11). This problem was discovered in the Irish *Superbrain* series of questions. Using Pythagoras' theorem one can calculate the lengths of sides of the blue and green triangles. It therefore turns out the green triangle is *similar* to the blue one, enlarged by scale factor $\sqrt{2}$. Hence the bottom left green triangle interior angle can be assigned to be β, which proves the sum of $\alpha + \beta = 45^o$.

4 Concluding thoughts

"In most sciences, one generation tears down what another has built, and what one establishes, another undoes. In mathematics alone each generation adds a new story to the old structure."
(Hermann Hankel, from Bayazit, 2009).

"Mathematics is a subject unlike any other. It has different definitions according to the context where it is used. Mathematics is seen as a culture of formal thinking; as a kind of mental activity, a social construction involving conjectures, proofs and refutation; and as concrete and mental representations of numbers, images, and objects related to differences, similarities, patterns, or regularities. Although it seems to be a quantitative science due to its relationship with numbers, mathematics differs from quantitative sciences by deductive reasoning. Each mathematical object should be consistent with the rest of the system. It is at this point that proof comes into play. Any new addition to the system should go through the proving process to maintain consistency. Any proposed piece of mathematical information should be checked through deductive reasoning. This is one of the features that situates it at the center of mathematics." (Bayazit, 2009)

I agree wholeheartedly with de-Villiers, Tall, Hersh and many others that mathematics is the art of explanation, and that appropriately chosen proofs can illuminate the beauty and power of the subject to students at all levels. Although my pre-teaching life as a physicist and engineer has typically led me to present mathematical ideas in a physical context, the increasingly proof-driven pedagogy of the Winchester teaching experience of the last few years has enabled me to reconnect with a fundamental curiosity and desire to answer the 'why' questions of pure mathematics. I certainly intend to continue this project, and aspire to bring step-by-step clarity to every idea I discuss with my students.

5 Glossary

The following mathematically-related definitions are taken from the *Oxford English Dictionary* (2005).

Conjecture

1. An opinion or conclusion formed on the basis of incomplete information.
2. An unproven mathematical or scientific theorem.

Origin: Late Middle English: (in the sense 'to divide' and 'divination'): from Old French, or from Latin *conjectura*, from *conicere* 'put together in thought', from *con* -'together' + *jacere* 'throw.'

Epistemology

1. Theory of knowledge, especially with regards to its methods, validity and scope, and the distinction between justified belief and opinion.

Origin: Mid 19^{th} century: from Greek *episteme* 'knowledge', from *epistasthai* 'know, know how to do'.

Heuristics

1. Enabling a person to discover or learn something for themselves.
2. Proceeding to a solution by trial and error or by rules that are only loosely defined.

Origin: Early 19^{th} century: formed irregularly from Greek *heuriskein* 'find'.

Lemma

1. A subsidiary or intermediate theorem in a anrgument or proof.
2. A heading indicating the subject or argument of a literary composition or annotation.

Origin: Late 16^{th} century: via Latin from Greek *lemma* 'something assumed'; related to *lambanein* 'take'.

Proof

1. Evidence or argument establishing a fact or the truth of a statement.
2. A series of stages in the resolution of a mathematical problem.

Origin: Latin *probare* 'to test'; Late latin *proba*; Old French *proeve*; Middle English *preve.*

Theorem

1. A general proposition not self-evident but proved by a chain of reasoning; a truth established by means of accepted truths.
2. A rule in algebra or other branches of mathematics expressed by symbols or formulae.

Origin: Mid 16th century: from French *théorème*, or via late Latin from Greek *theōrema* 'speculation, proposition', from *theōrein* 'look at', from *theōros* 'spectator.'

6 References

1. Alcock, L. (2004). Uses of Example Objects in Proving. *Proc. 28th Conference of the International Group for the Psychoogy of Mathematics Education,* 2004. Vol. **2.** pp17-24.

2. Ball, D.L., Hoyles, C., Jahnke, H.N., Movshovitz-Hadar, N. (2002). The Teaching of Proof. ICM 2002. Vol III 1-3. pp907-920.

3. Bayazit, N. (2009). Prospective Mathematics Teacher's Use of Mathematical Definitions in Doing Proof. PhD dissertation. Florida State University, College of Education.

4. Benos, Dale J. *et al.* (2007). "The Ups and Downs of Peer Review". *Advances in Physiology Education* **31** (2): 145–152. doi:10.1152/advan.00104.2006. PMID 17562902. pp145-

5. Bogomolny, A., *What Is Proof? from Interactive Mathematics Miscellany and Puzzles.* http://www.cut-the-knot.org/WhatIs/WhatIsProof.shtml. Accessed 17 December 2013

6. Brown, M., Jones, K., Taylor, R., Hirst, A. (2004). *Proc. 27th Annual Conference of the Mathematics Education Research Group of Australasia (MERGA27), 27-30 June 2004, Townsville, Queensland, Australia.* Vol **1**, 127-134.

7. Cadwallader-Olsker, T. (2011). What do we Mean by Mathematical Proof? *Journal of Humanistic Mathematics.* Vol **1**. No. 1. pp33-60

8. Chambers, P. (1999), Teaching Pythagoras' theorem. *Mathematics in School.* Vol. **28**. No 4. pp. 22-24.

9. de-Villiers, M. (1999). *Rethinking Proof with Sketchpad.* Key Curriculum Press.

10. de Villiers, M. (2012). An illustration of the explanatory and discovery functions of proof. http://dx.doi.org/10.4102/pythagoras.v33i3.193. Accessed 18/12/13.

11. *Geometric Proof: Nuffield Foundation Key Ideas.* http://www.nuffieldfoundation.org/key-ideas-teaching-mathematics/geometric-proof. Accessed 18/12/13.

12. *Gödel's Incompleteness Theorems.* Stanford Encyclopedia of Philosophy. http://plato.stanford.edu/entries/goedel-incompleteness/. Accessed 19/12/13.

13. Hanna, G. (2000). Proof, Explanation and Exploration: An overview. *Educational Studies in Mathematics* Vol. **44**: pp 5-23.

14. Harel, G., Sowder, L. (1998). Students' Proof Schemes: results from Exploratory Studies. *CBMS Issues in Mathematics Education.* Vol. **7**.

15. Healy, L., Hoyles, C. (2000). A Study of proof Conceptions in Algebra. *Journal for Research in Mathematics Education.* Vol. **31**, No.4, pp396-428.

16. Hersh, R. (1993). *Educational Studies in Mathematics.* Vol **24**: pp389-399.

17. Hoyles, C., Jones, K. (1998), Proof in Dynamic Geometry Contexts. *Perspectives on the Teaching of Geometry for the 21st Century.* Dordrecht: Kluwer. pp121-128. ISBN: 0792349903

18. *Interpreting the Mathematics Curriculum: Developing reasoning through algebra and geometry.* (2004). UK Qualifications and Curriculum Authority.

19. Inglis, M. (2009). The Effect of Authority on the Persuasiveness of Mathematical Arguments. *Cognition and Instruction.* Vol. **27**, Issue 1.

20. Jaffe, A., Quinn, F. (1993). "Theoretical Mathematics" : Toward a Cultural Synthesis of mathematics and Theoretical Physics". *Bulletin (New Series) of the American Mathematical Society.* Vol. **29**, No. 1.

21. Jones, K. (1997). Student teachers' conceptions of mathematical proof. *Mathematics Education Review*, Vol. **9**, pp21-32.

22. Jones, K. (2000). The Student Experience of Mathematical Proof and University Level. *International Journal of Mathematics Education, Science and Technology.* Vol. **31**, No. 1. pp53-60.

23. Knuth, E.J. (2002). Teacher's Conceptions of Proof in the Context of Secondary School Mathematics. *Journal of Mathematics Teacher Education.* Vol. **5**: pp61–88.

24. Knuth, E. J., Slaughter, M., Choppin, J., & Sutherland, J. (2002). Mapping theconceptual terrain of middle school students' competencies in justifying andproving. *Proceedings of the 24th Meeting for PME-NA.* Vol. **4**, (pp.1693-1700). Athens, GA.

25. Leikin, R., Grossman, D. (2013). *Educ. Stud. Math.* Vol. **82**. pp 515–531

26. Lockhart, P. (2002). *A Mathematician's Lament.*
`http://www.maa.org/external_archive/devlin/devlin_03_08.html`

27. Maher, C., Martino, A.M. (1996). The Development of the Idea of Mathematical Proof: A 5-Year Case Study. *Journal for Research in Mathematics Education.* Vol. 27, No. 2 (Mar., 1996), pp 194-214

28. Maher, C. (2005). How Students Structure their Investigations and Learn Mathematics: Insights from a Long-Term Study. *Journal of Mathematical Behavior.* Vol. **24.** pp1–14.

29. NRICH Pythagoras Proofs. `http://nrich.maths.org/6553.` Accessed 18/12/13.

30. Nyaumwe, L., Buzuzi, G. (2007). Teacher's Attitudes Towards Proof of Mathematical Results in the Secondary School Curriculum: the Case of Zimbabwe. *Mathematics Education Research Journal.* Vol. **19**, No. 3, 21–32.

31. *Oxford Dictionary of English* (2005). 2$^{\text{nd}}$ Edition (Revised). Oxford University Press.

32. Peressini, D., Borko, H., Romagnano, L., Knuth, E., Willis, C. (2004). A Conceptual Framework for Learning to Teach Secondary Mathematics: A Situative Perspective. *Educational Studies in Mathematics.* Vol. **56**: 67–96.

33. Polya, G. (1957). *How To Solve It.* 2$^{\text{nd}}$ Edition. Princeton University Press, 1957, ISBN 0-691-08097-6. Summary at `http://math.berkeley.edu/~gmelvin/polya.pdf.` Accessed 18/12/13.

34. Stylianides, G., Stylianides, A.J. (2006). "Making Proof Central to Pre-High School Mathematics is an Appropriate Instructional Goal: Provable, Refutable, or Undecidable Proposition?" *Proc. 30$^{\text{th}}$ Conference of the International Group for the Psychology of Mathematics Education.* Vol. **5**, pp. 209-216.

35. Tall, D. (1989). The Nature of Mathematical Proof. *Mathematics Teaching,* Vol. **127**, pp28–32.

36. Tall, D. (1994). "Understanding the Process of Advanced Mathematical Thinking." *Transcript of an invited ICMI lecture at the International Congress of Mathematicians, Zurich.*

37. Tall, D. (2002). Differing Modes of Proof and Belief in Mathematics. *Proc. International Conference onMathematics: Understanding Proving and Proving to Understand.* pp 91–107. National Taiwan Normal University, Taipei, Taiwan.

38. Tall, D., Mejia-Ramos, J.P. (2006). The Long-Term Cognitive Development of Different Types of Reasoning and Proof. *Conference on Explanation and Proof in Mathematics: Philosophical and Educational Perspectives*, Essen, Germany.

`http://homepages.warwick.ac.uk/staff/David.Tall/themes/three-worlds.html`

39. Thurston, W.P. (1994). On Proof and Progress in Mathematics. *Bulletin of the American Mathematical Society.* Vol. **30**, No. 2, April 1994, pp 161-177.

40. Weber, K., Brophy, A. & Lin, K. (2007). Learning Advanced Mathematical Concepts byReading Text. *Proceedings of Research in Undergraduate Mathematics Education*, SanDiego, CA.

41. Zack, V. (1997). 'You Have to Prove us Wrong': Proof at the Elementary School Level. *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education.* Vol. **4**, pp. 291-298. Lahti, University of Helsinki.