Schools Direct Assignment 2
Intervention for Learning: Analysis & Evaluation
Teaching Reasoning and Proof

Dr Andrew French

April 27, 2014

1 Summary

This assignment continues from Assignment 1 with an analysis of the teaching of mathematical reasoning and proof in secondary schools. Whereas Assignment 1 focussed upon a review of literature, and a reflective analysis of my own professional practice, this assignment considers the impact of a specific 'proof driven' intervention in two of my classes. The topic is the teaching of Circle Theorems at IGCSE level and the classes comprise middle to relatively high ability Year 9 and Year 10 students (sets five and three out of eight accordingly).

The assignment is split into three sections. The first is the Intervention Plan, which describes the scope of the intervention, and the literature-driven motivation from which the requirements of the design are derived. 'Design Research' is the dominant methodology. The second section is an analysis of the observations recorded during the intervention. A summary of what happened during the intervention, based upon my diary entries and work done by the students in and outside the classroom, is included in the Appendix. Lastly, the third section evaluates the intervention and contains a personal reflection on how the work might influence my teaching practice in the future.

2 Intervention plan

2.1 Introduction

2.1.1 The research question

- **"How effective is an overt proof-based design for a unit of work based on the teaching of Circle Theorems?"**
  - What 'proof schemes' do students naturally adopt?
  - Can they be taught to think in a more analytical fashion? Can they learn to appreciate the problems inherent with *External Conviction* or *Empirical* proof schemes? (Harel & Sowder, 1998). Can they be taught to recognise that they might be thinking this way without being initially conscious of it?
  - Can pupils be motivated to explore, develop and *remember* proofs?

2.1.2 Scope and limitations

This assignment is intended as a 'design research' based analysis of a two-week unit of work relating to the teaching of Circle Theorems. The level is higher-tier IGCSE and a similar set of basic resources will be used for two classes, A and B (although the anticipated pace and opportunity for discussion is anticipated to differ significantly between the classes). Class A is a group of twenty relatively bright Year 9 boys. This will be their first experience of Circle Theorems, and the unit of work will follow an introduction to basic geometry involving parallel lines, polygons and symmetry. Class B is a similarly sized group of boys in Year 10. They are, for their year, in a slightly higher set than Class A, and will have met Circle Theorems before.

2.1.3 Summary of Assignment 1 literature review

In Assignment 1, the Literature Review suggested the following main points:

1. Students tend to adopt *empirical* "pattern spotting" proof schemes (or worse, *authoritarian* ones). i.e. students might be satisfied that a theorem is true based upon a few confirming examples, or simply because their teacher stated a mathematical result as a fact. [Harel & Sowder (1998), Jones (1997), Ball *et al* (2002), Weber *et al* (2007)].

2. Communication between research mathematicians can be "dysfunctional" [Tall (1994) and Thurston (1994)] and this may lead to a cascade of confusing messages regarding the nature and importance of mathematical proof at secondary school level. (Jones, 2000).

3. There is no international consistency in the teaching of mathematical proof. Often it is "peripheral" and typically restricted to geometry (e.g. the US) or over-intertwined with empirical discovery-learning (UK, Holland). [Knuth, (2002), Cadwallader-Olsker (2011), Hanna (2000), Nyaumwe *et al* (2007), Stylianides & Stylianides (2006), Ball *et al* (2002)].

4. Although the principal function of mathematical proof is as a verification mechanism of theorems (for example, *Pythagoras' Theorem* can be *deduced* to be true from more basic axioms), de-Villiers and others argue that there are many other important aspects, mostly relating to proof as a vehicle for the communication of mathematical ideas and the clear explanation of mathematical results. [de-Villiers (2012), Alcock (2004), Ball *et al* (2002), Hanna (2000)].

2.1.4 Intervention requirements

The conclusions of Assignment 1 lead me to define the following requirements for the design of my unit of work:

1. The reason *why* we are proving Circle Theorems must be very clearly stated. The idea is to communicate what *axiomatic deduction* says about Mathematics and mathematical reasoning, rather than just a verification that a theorem is true.

2. The *order* of Circle Theorems presented is set by the natural pedagogy driven by systematic proof. 'Later' results (e.g. the *Alternate Segment Theorem*) will make use of 'previous' Circle Theorems such as the *Arrowhead Theorem*. The students will therefore be exposed to a *system* of interconnected proofs, exemplific of the structure of Mathematics as a whole.

3. Clarity and simplicity must be paramount. All mathematical tools will be blended appropriately. Algebra, diagrams, use of colour, numerical examples etc will be employed as appropriate to maximise explanatory power. Clarity will be much more important than brevity.

4. Natural language will be used and mathematical jargon minimised. (e.g. *Arrowhead Theorem, Mountain Theorem*). See Appendix for precise definitions of these theorems and associated visual resources.

5. Worksheets and tests will be designed to facilitate the use of proof. Although there will certainly be some managed class discussion, students will be introduced step-by-step to recommended proofs rather than asked to somehow 'come up with their own.' In other words, proof elements will be integrated into my department's normal teaching pattern of:

    - Class discussion of a new idea or problem
    - Example problems done together
    - Example problems done by students singly or in pairs with varying degrees of teacher help
    - Problems done individually by students for homework
    - Revision via class discussion and further example problems

- Test
- Incorporation of methods into another idea or topic as appropriate
- Repetition of pattern in 'learning spiral', continuously embedding previous ideas using homework ('outers') as the primary vehicle

2.2 Research methodology

2.2.1 Scope of research method

The intervention will proceed as a piece of 'design research,' i.e. an analysis of the practical efficacy of a designed unit of work.

The work will be limited in scope to my Year 9 and Year 10 classes (of boys). Observations and conclusions will be reported in this context, and any more general inferences will be appropriately caveated.

Given the small sample size (approximately forty boys in total) and short duration of the intervention (about two weeks), significant statistical analysis will be avoided since it will almost certainly be numerically invalid. Instead, evidence for and against a hypothesis such as 'do students naturally adopt an empirical proof scheme?' will be compared in a structured, but qualitative, manner.

2.2.2 Criticism and theoretical underpinning of chosen research method

Although the proposed intervention of using overt proof-driven pedagogy in a unit of work is, according to Knuth, (2002), possibly *unusual* in the modern (Western) classroom, the research method proposed is a qualitative, descriptive study based upon observations and reflections of *typical* classroom practice rather than a controlled 'experiment.' Bell (2005) and Clarke (2005) refer to this approach as a genuine model of research and not just a 'soft option' compared to a more quantitative, comparative study, as long as any inferences drawn are suitably contextualised. Alternative summaries and discussion of research methodologies were also consulted: Hammersley (2012), Rajasekar *et al* (2013), Bradford School of Management *Introduction to Research and Research methods*, Centre for Local Economic Strategies (CLES) *Research Methods Handbook* (2009), plus online sources such as the comprehensive Plymouth University Research in Education (RESINED) website. The latter would perhaps describe my research as a 'Case Study' (i.e. restricted to a specific situation rather than an attempt at idealisation or generality) whereas Clarke (2005), and also Wollard (website) would perhaps describe it as 'Action Research,' (i.e. an analysis of a 'real world' and not laboratory based scenario). Regardless of label, the key caveats to any conclusions drawn for this type of study are as follows:

1. Since the research is to proceed in a normal classroom environment, no 'control group' is monitored who will receive a 'typical' education on Circle Theorems. Therefore a comparative analysis is not possible. This would be hard in any case, as 'typical' is not easy to define since every teacher will have their own style of exposition at this level of detail. Also, every child is behaviourally and cognitively different and has their own learning needs and styles. A comparative study is almost certainly flawed unless a statistically significant number of classes are studied (and divided into intervention and control groups) such that the variations between teachers and children are averaged out. For this to work the variation itself would have to be quantified and measured which would in turn magnify tremendously the complexity of the study!

2. Identifying the proof scheme adopted by students, *is* possibly a discernible metric for comparison of my observations and those reported in the literature. [e.g. Harel & Sowder (1998)]. Although observations of two classes at different stages and ability levels will enrich the findings, they will still be germane to the specific educational environment of the school that I teach at.

3. As a teacher I have a vested interest in the intervention to work and will therefore risk adding self-confirmation bias to any evaluation. In delivering the intervention I am therefore part of the system and hence cannot be an independent observer. The learning of the students is clearly influenced by my physical presence and presentation style as well as any written materials I produce. To counter these problems I will endeavour to report factual observations of what happened in the classroom, and use homework tasks and tests (i.e. independent of my direct interaction) to augment my on-the-fly assessment of learning.

2.2.3 Materials

Materials to be generated will be:

1. Lesson plans (at least one of the lessons during the intervention will be formally observed)
2. Handouts, defining all terms and summarizing the discussion that will occur with the whole class. These will be the core reference material. Handouts will be given to the students *after* a group discussion to maximise attentiveness during class. The students will be encouraged to make their own notes during the initial discussion.
3. Worksheets which contain classic Circle Theorem problems, plus solutions.
4. Special 'proof' worksheets which include multi-step problems which relate to the steps in a proof. The goal is for the student to make a general connection from the workings of a specific problem.
5. Homework problems with 'proof' questions.
6. Homework problems which ask students to write their thoughts on what they think mathematical proof means.
7. A test of Circle Theorem problems, with proof elements.
8. A memory test of Circle Theorem proofs (with suitable scaffolding and hints).

2.3 Ethical considerations

This study will be conducted in a normal two-week teaching period and will not require any additional contact time with students. No interviews will be conducted. Although the content of students' work will be scanned during this period to facilitate analysis, any conclusions will be anonymised. Classes will be referred to as A and B, and students will be referred to by numbers (1,2,3....). There will be no systematic use of numbers, so it will not be possible for a student's name to be inferred from a particular number.

All students will be briefed by me that I will be conducting this study, and that the design of the unit of work is an 'experiment.' The proof elements will augment, rather than replace normal learning objectives. In other words, all students will receive basic knowledge as dictated by the departmental scheme of work and will therefore not be penalised compared to other classes that are not being exposed to a 'proof driven pedagogy' experiment.

3 Analysis

3.1 Summary of intervention

The intervention between the 8th and 28th of January 2014 followed the following high level pattern for both classes:

1. Introduction to the idea of mathematical proof, and how this is different from what can be achieved with the scientific method.
2. Foundations of proof in Geometry. (First time for class A and a breezy revision for class B). 'Z-angles,' angles in a triangle sum to 180^o etc.
3. Fairly traditional on-the-board active discussion of Circle Theorem proofs, starting with the *Arrowhead Theorem*. Further results are always prompted by "what do we already know? How can we change the diagram in such a way that we can make use of these facts?" (e.g. split the Arrowhead into triangles, and apply what we know about the internal angles of triangles to prove the *Arrowhead Theorem*).
4. Consolidation using problems.

5. Homework essay question (suitably scaffolded with examples) designed to capture the students' conception of proof.

6. Knowledge of Circle Theorem proofs test.

7. Circle Theorem problems test.

8. Washup and revision of what has been learned.

Overall, a high level of engagement was consistently noted for both classes. (See Reflective Diary entries in the Appendix). I also received very positive feedback from a senior colleague who conducted a formal observation during the intervention.

3.2 Structure of analysis

To structure the analysis let us return to the research question: **"How effective is an overt proof-based design for a unit of work based on the teaching of Circle Theorems?"** I shall comment on my observations under the following sub-headings:

- What 'proof schemes' do students naturally adopt?
- Analysis of students' learning. Can they be taught to think in a more analytical fashion? Can they learn to appreciate the problems inherent with *External Conviction* or *Empirical* proof schemes? (Harel & Sowder, 1998). Can they be taught to recognise that they might be thinking this way without being initially conscious of it?
- Can pupils be motivated to explore, develop and *remember* proofs?

The tables in figures 1 & 2 below summarise the responses to the essay question: "Write, using some examples, what you think PROOF means in Mathematics." Examples of students' writing from each class are presented in figures 3 & 4.

Class A

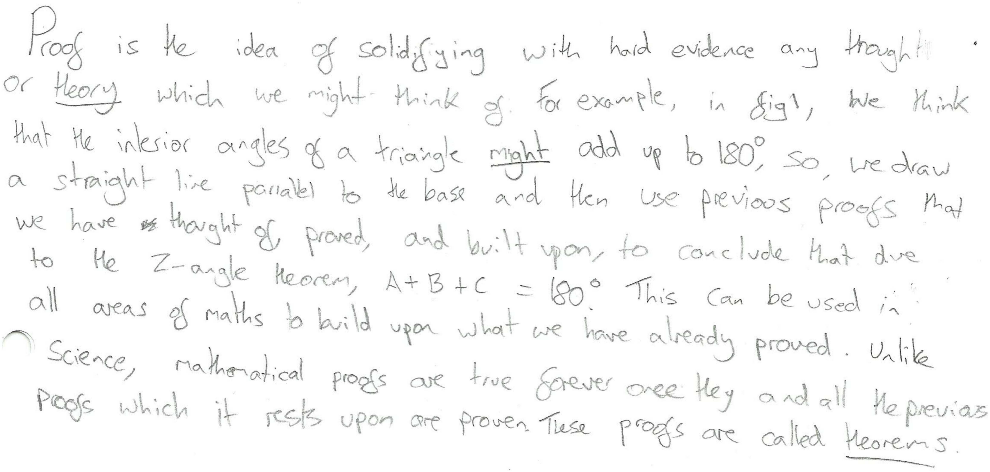
Use of mathematical equations to show something is correct.
A set of Mathematical evidence that shows something is true. Using something you already know to show that something else is correct.
A full explanation for why some things happen, not just taking them as read.
Proof in Mathematics means that if you (can) find the answer, it is how you show that it is correct.
Proof means understanding of how something is and how it became to be what it is.
The way in which we show how a theorem is correct.
A way Mathematicians can see why things are the way they are.
Proof in Mathematics is when you have a theorem and you need to make somebody believe it is right.
Proof means when you show something is true in Mathematics.
Proof is when there is a theory or conjecture that works on any occasion.
The idea of solidifying with hard evidence any thought or theory which we might think of.... Can be used in all areas of maths to build upon what we have already proved. Unlike Science, mathematical proofs are true forever once they and all the previous proofs which they rest upon are proven.
Proof is about proving a theory and actually giving yourself a confirmed reason to trust in it.
The use of Mathematics to 'prove' a law or theorem of maths by explaining how it is correct.
A theorem is the knowledge that whatever the input, a certain answer will always be obtained. You can find a proof in many different ways but it will always be a set of instructions that conclusively show that the theorem will work with any input.
To quote my dictionary, proof is "evidence which establishes that something is true." The reason one proves a concept in Mathematics is to find the simplest ways of solving problems.
The use of diagrams to explain a theorem.
The ability of being able to fully explain a theory by putting it into practice via examples, the results of which successfully prove right the conjecture. Proof is about putting theories to the test. Proof is being able to verify a theorem by testing it against other theorems, and using numbers instead of unknowns.

Figure 1: Summary of responses from class A to homework essay question "Write, using some examples, what you think PROOF means in Mathematics."

Class B

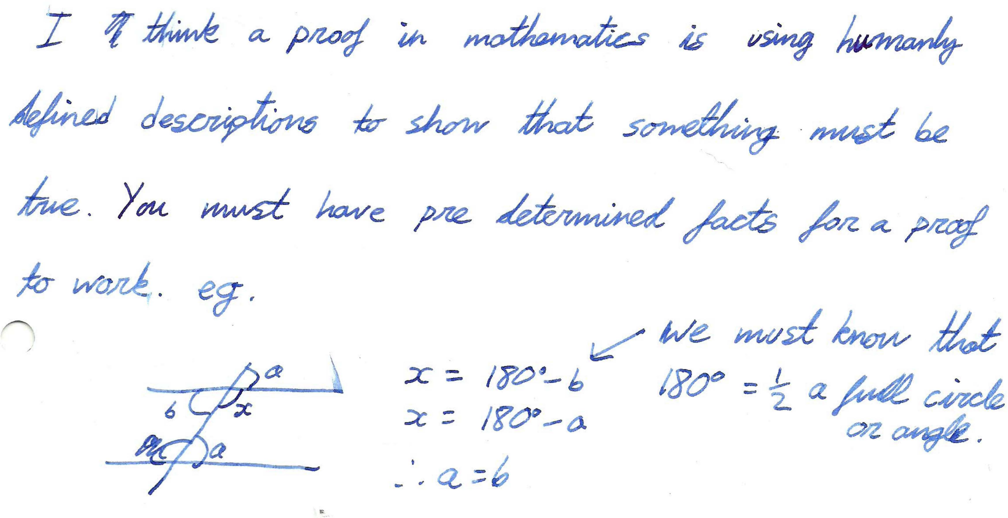
Proof means to prove how a theory/rule/formula works.
A way of showing something is true. Verifying something so just to double check that it is right.
The breakdown of a method or an answer that is widely used. Explaining why a method works and how it works.
How you show that a pattern you have noticed is always applicable and therefore a rule.
Evidence that something is true.
In mathematics it is possible, unlike science, to prove that something is definitely, absolutely, without a doubt, correct. This is because we can apply a theory to all possible situations in hypothetical mathematics, rather than physically in science.
Taking an expression or diagram and showing that it is mathematically sound by applying simpler facts to it.
Using numbers and undeniable evidence to justify a theory, or even disprove one.
A way of combining known facts and rules to explain why something is the way it is.
Using humanly defined descriptions to show that something must be true. You must have pre-determined facts for a proof to work.
The theoretical testing of formulae to see whether or not they will be correct in all and every circumstance and variation.
Showing evidence for mathematical calculations.
The group of workings relying on universally accepted and proven mathematical laws to show a mathematical theory works. This is generally done by assuming the theory is false.
Most commonly an algebraic statement which proves a theory or equation.
Explaining how you get the answer. Saying why the answer is right. Using theorems to get answers.
An argument for a mathematical statement. Previous mathematical facts can be used to help show that the theorem in question is in fact true.
The explanation of why (a) certain short-cut method works. The proof is essentially a long way round that has been simplified and shortened.
Proves something specific through a specific equation or diagram and not trial and error.

Figure 2: Summary of responses from class B to homework essay question "Write, using some examples, what you think PROOF means in Mathematics."



Proof is the idea of solidifying with hard evidence any thought or theory which we might think of. For example, in fig1, we think that the interior angles of a triangle might add up to 180°, so, we draw a straight line parallel to the base and then use previous proofs that we have thought of, proved, and built upon, to conclude that due to the Z-angle theorem, A + B + C = 180°. This can be used in all areas of maths to build upon what we have already proved. Unlike Science, mathematical proofs are true forever once they and all the previous proofs which it rests upon are proven. These proofs are called theorems.

Figure 3: A (class A) student's response to the question "Write, using some examples, what you think PROOF means in Mathematics."



I think a proof in mathematics is using humanly defined descriptions to show that something must be true. You must have pre determined facts for a proof to work. eg.

$x = 180° - b$
$x = 180° - a$
$\therefore a = b$

We must know that 180° = ½ a full circle or angle.

Figure 4: A (class B) student's response to the question "Write, using some examples, what you think PROOF means in Mathematics."

3.3 Students' proof schemes

Harel & Sowder (1998) propose three *Proof Schemes* as a model for the concept of mathematical proof as held by students. [Note the explanations below are my own, and were written as part of Assignment 1. They are included here for ease of reference].

1. **External conviction proof schemes**
   (a) *Ritualistic* e.g. "The area of a circle is pi r squared, because it is (and we chanted this fact aloud in class until we could remember it)."
   (b) *Authoritarian* e.g. "Because our teacher told us it was true".
   (c) *Symbolic* e.g. "Taking the square root removes the little two from the index of a number, so $\sqrt{a^2+b^2} = a + b$, right?"
2. **Empirical proof schemes**
   (a) *Inductive* e.g. "this formula seems to hold for all the terms I have checked so far, therefore it must be true."
   (b) *Perceptual* e.g. "this looks like it could be a right angle. Therefore it is."
3. **Analytical proof schemes**
   (a) *Transformational* e.g. "I dragged the vertex of the circumscribed arrowhead around and the ratio between angles $A\widehat{B}C$ and $A\widehat{O}C$ remained the same!"
   (b) *Axiomatic* e.g. "The sum of the internal angles of a triangle is 180^o. I can divide up a pentagon into exactly three non-overlapping triangles, so therefore the sum of the internal angles in a pentagon must be $3 \times 180^o = 540^o$"

3.4 What 'proof schemes' do students naturally adopt?

The responses from both classes show a high uniformity of definitions of proof corresponding to the idea of a *test for truth* "in all and every circumstance and variation." [Class B response #11]. This would align with the conventional *verification* aspect of mathematical proof [Hersh (1993), Hanna (2000), de-Villiers (2012)]. Many pupils talk about the requirement to explain the mechanism of proof, i.e. an argument which shows *why* a result is true. Several students allude to the structural feature of a proof as being based upon previously held truths. If this overally philosophy is retained by the children then it implies they have the potential to adopt *Analytical* proof schemes, and not be swayed by the fallacies of *External Conviction* or *Empirical* schemes. However, let us not get too carried away. For most of the students, this an initial introduction to the idea of mathematical proof, and their thinking (hopefully!) will have been conditioned by my teaching. An exploration of a proof of a different type of theorem (perhaps algebra) later on in the year, without much help from me initially, might be an insightful follow up. [See Evaluation section below].

3.5 Methodological criticism

A valid criticism of my study is that the essay question could have been set prior to any discussion of proof. The analysis is perhaps flawed in the sense that one hopes my teaching will have modified what the students 'naturally' adopt. But then again, what is a 'natural' state? The students have been learning since they were born and to deny them any form of mathematical training would certainly be unethical! In this context the 'natural state' is really their *prior* state, and that will be different for all students since they were educated in different places and by different people. The idea of a 'natural state' has therefore limited practical relevance, unless one takes it to mean the *typically* held proof schemes (in a statistical sense of the overall student population) as alluded to by Harel & Sowder.

3.6 Analysis of students' learning

The observations above are essentially an analysis of the internalisation of the learning associated with the idea of mathematical proof. The ideas expounded by the students appear to align with quite high levels in Knowledge (*Cognitive*) domains in Bloom's Taxonomy. I would rate responses between *Apply* and *Evaluate*. (Bloom & Krathwohl, 1956). In terms of position in the 'Kolbian learning cycle' (Kolb, 1981; Kolb, 1984; Kolb & Kolb 2005, Honey & Mumford, 1982, Mainemelis *et al*, 2000; McLeod, 2010) the activity moved the students from *Activist* to *Reflector* roles, and perhaps *Theorist*. Note the model of Jarvis (1987) is perhaps a more realistic idealisation of the learning process of the students, but in essence uses the same terminology. To complete the Kolbian cycle (and to attain the highest *Synthesise*, *Create* levels of Bloom's *Cognitive* domain) the students would have to apply their new skills to new problems. To a certain extent the application of their knowledge of proof was assessed using the tests (see below), but I would suggest a fresh and lightly scaffolded approach to a completely new topic would be a much more rigorous test of their abilities.

3.7 Can pupils be motivated to explore, develop and *remember* proofs?

My Reflective Diary entries (see Appendix) coresponding to the intervention record some of the highest levels of engagement I have had from my students all year. They were clearly intrigued by the idea of proof and all students were motivated to participate in the discussions which generated the theorems. (I managed the discussion to include a wide range of responses, not just from the most able and most extrovert). Although I led these sessions, all the results were effectively *created* following responses to questions posed to the class. I would like to believe the number of references to the mechanistic nature of mathematical proof in the students' essay responses is a positive indicator of their engagement with my teaching.

As discussed above, further research would be required to investigate whether the students could devise their own proofs effectively. However, I should report that (especially for class B), there were several instances of valid alternative steps in the proof of the *Alternate Segment Theorem* submitted as answers to the Proof test. Although the idea was to see if students could remember my recommended proof, the outcome was perhaps even better for the learning of the students (i.e. 'more advanced' in Bloom's *Cognitive* domain) as they were able to add their own creations to the ensuing discussion.

For a more quantitative assessment of the efficacy of the intervention, figure 5 summarises the results of Proof and Problem tests taken by both classes at the end of the intervention. More information (in the form of histograms and cumulative frequency graphs) is provided in Appendix 2. In all cases, the marks are some of the best I have collected all year, and certainly rank favourably with other topics. At the most basic level, I feel confident that the students were learning 'in the moment' and also able to retain and apply the new material. As expected, class B's scores are significantly higher than class A (an average deviation of around 20%). Specifically, class B marks for the proof test are very high indeed, and also the problem test results are both higher on average and more closely spread than those of class A. One may infer that class B pupils are more able to accept the concepts 'as is' since they have a wider and more sophisticated structure of prior knowledge than class A. In terms of Social Development Theory (Vygotsky, 1962), their *Zone of Proximal Development* is perhaps less wide for class B than class A. The proposed actions regarding future teaching of both classes based upon the results of this analysis are discussed in the Evaluation section below.

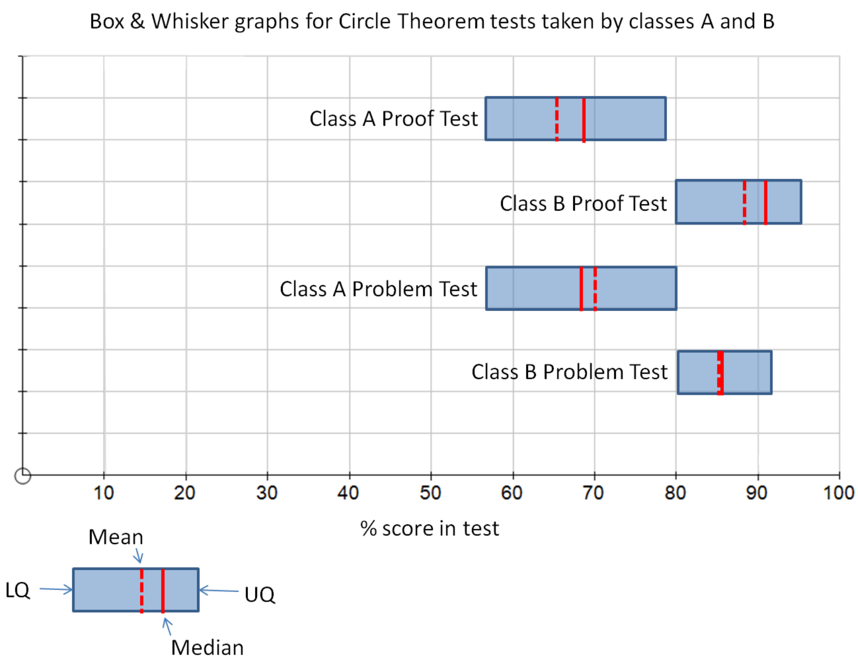


Figure 5: Box & Whisker plots which summarize the distribution of marks for the Proof and Problem tests taken by classes A and B following the conclusion of the Circle Theorem intervention. The same tests were given to both classes.

4 Evaluation

4.1 Assessment of the quality of learning by school students and an exploration of how learning might be improved in future teaching

As discussed in the previous section, I think I can justify effective learning was occurring for both classes during the intervention for the following reasons:

1. Reflective diary entries describe high levels of engagement
2. Essay question responses indicate a retention and internalisation of the key concept of mathematical proof. All students were able to describe these ideas in their own words. Some responses were pleasingly sophisticated!
3. Proof test results indicate most students could retain knowledge of how to go about proving a Circle Theorem. The near flawless results of class B were, as expected, not matched by class A. Hence further consolidation is required for this class. (Although the statistics don't show this, the key reason is that proofs of the *Alternate Segment Theorem* for class A were much more sparse than those of class B. The marks for the introductory questions, which focused on proving the internal angle of a triangle sum to 180^o and *Arrowhead, Mountain* and *Right Angle* theorems, were uniformly high for both classes).
4. Both classes scored acceptably high marks on the Problem test. Again, the median and inter-quartile range for class A were, respectively, lower and wider than class B. This indicates some students in class A require significant further consolidation of the topic. This will certainly happen in the form of homework questions and ultimately another visit to Circle Theorems next year (and again when they prepare for their IGCSE exams in Year 11).
5. I received favourable feedback from the deputy head of Mathematics, who observed one of my lessons. Indeed, the 'proof driven pedagogy' initiative may well influence what is deemed recommended practice by the school.

I think several aspects could be improved as these were not explored in the study. Actions for a future study could be:

1. Setting the 'what do you think...' essay question *before* any discussion of proof might serve as an alternative means of gauging prior knowledge (rather than a check of knowledge internalisation as was used in this intervention). Ideally I should have set the same question before and after the intervention.
2. Use of some form of experimentation stage, perhaps using geometric software, might prove effective for conjecture generation. [de-Villiers, 1999; Hoyles *et al*, 1998; Leikin, 2013, and also Polya (1957) who suggested that Mathematicians only bother to pursue a rigorous proof once they have developed a belief that a proposition is likely to be true].

4.2 Impact of the intervention or study on my own classroom practice

As mentioned above, the idea of a 'proof driven pedagogy' seems to have worked as a concept. The structure of the proof arguments, based on the recommendations and models of de-Villiers (2012), Cadwallader-Olsker (2011) and Hanna (2000) enabled me to create a series of lessons (and associated material) which motivated the students. Although this is a very limited case study, I have shown it is possible to mitigate the educational fears of Hersh (1993), Ball *et al* (2002), Jones (1997,2000), Knuth (2002) and Stylianides & Stylianides (2006), in at least in two classes of one rather unique educational establishment! I shall certainly endeavour to add more proof elements to other topics. In departmental meetings we have already discussed candidate topics such as basic number theory and formulae for the area and volume of basic shapes and solids.

The free-text essay question also yielded surprisingly useful insights. Occasional use of this concept in homework tasks could prove a fruitful modification to my normal teaching habits.

4.3 Recommendations for future work

As discussed above, an investigation of the efficacy of a proof driven pedagogy thread inherent in the totality of my Mathematics teaching would be an extension of this study. In practical terms, the Circle Theorems topic could be expanded to a modest selection of other syllabus items as discussed above. A key aspect would be to re-assess the proof-scheme (Harel & Sowder, 1998) adopted by students at a later stage to see if they revert from the *Analytical* to more flawed *Empirical* or *Authoritarian* schemes. A candidate topic might be formulae for number sequences, as there is a great temptation here to confuse pattern-spotting with a rigorous proof. Students who study Further Mathematics in Year 12 will certainly encounter *Proof by Induction*. Perhaps the ideas behind this method could be introduced to brighter students (e.g. those of class B ability) earlier, perhaps in Year 11. A revisit of Healy & Hoyles' work of students' proof conceptions in algebra (2000) might be sensible if this approach is adopted.

4.4 Personal reflection on the work undertaken, its limitations and its impact on my professional development as a teacher

I began this study motivated by a personal belief that mathematical proof should form a part of my teaching. It is inherently a creative activity, and at best can encapsulate what it truly is to do Mathematics. I felt liberated by the calling of de-Villiers (2012), Hanna (2000) and others for the *explanatory* nature of proof to be highlighted, rather than proof being only associated with a small cadre of very erudite persons ensconced in the Ivory Tower of Research Mathematics, who can *verify* a theorem is true in a rigorous but largely incomprehensible manner. It is not enough for me to just use a result. I need to understand *why* it is true on an intuitive level using clear and simple arguments. I think many students will share this view, or at least can be encouraged to hold it temporarily.

Proving every mathematical result from first principles could easily become a burden and take the lightness out of my teaching, which is a particular aspect of my approach that I am striving to improve. However, saying to students "this result *can* be proved, this is the general argument, but there are techniques required (such as Calculus) which you will meet in a few years and so we won't go into the details of today" is very different from simply stating a result and then using it without inviting any further discussion.

The success of this intervention also helps to justify and shape my wider project of creating self-contained handouts for each topic that I teach. In addition to 'nice' worked examples which form the bulk of the pieces, *why a result is true* is also a fundamental component. A growing number of handouts is available online at `http://www.eclecticon.info/maths.htm`.

5 References

1. Alcock, L. (2004). "Use of Example Objects in Proving." In: Hoines, M.J. & Fuglestad, A.B. (*Eds*). *Proc. 28th*$^{\text{th}}$ *Conference of the International Group for the Psychology of Mathematics Education*, 2004. Vol **2** pp17-24.

2. Ball, D.L., Hoyles, C., Jahnke, H.N., Movshovitz-Hadar, N. (2002). "The Teaching of Proof." ICM 2002. Vol III 1-3. pp907-920.

3. Bell, J. (2005). *Doing your research project. A guide for first-time researchers in education, health and social science.* Fourth Edition. Open University Press. McGraw-Hill Education.

4. Bloom, B. S. & Krathwohl, D.R. (1956). *Taxonomy of educational objectives: The classification of educational goals, by a committee of college and university examiners. Handbook 1: Cognitive domain.* New York, Longmans. (Current copyright: Allyn & Bacon, Boston USA).

5. Business-Balls website - Kolb Learning Styles.

   `http://www.businessballs.com/kolblearningstyles.htm.` Accessed 29/1/14.

6. Cadwallader-Olsker, T. (2011). "What do we Mean by Mathematical Proof?" *Journal of Humanistic Mathematics.* Vol **1**. No. 1. pp33-60

7. Clarke, R.J. (2005). *Research Methodologies.* HDR Seminar Series. Faculty of Commerce. Spring Session 2005.

8. de-Villiers, M. (1999). *Rethinking Proof with Sketchpad.* Key Curriculum Press.

   `http://www.keycurriculum.com/`

9. de Villiers, M. (2012). "An illustration of the explanatory and discovery functions of proof." *Pythagoras*, **33** (3).

   `http://dx.doi.org/10.4102/pythagoras.v33i3.193.`

   Accessed 18/12/13.

10. Hammersley, M. (2012). "Methodological Paradigms in Educational Research." British Educational Research Association (BERA) on-line resource.

    `http://www.bera.ac.uk/resources/methodological-paradigms-educational-research`

    Accessed 20/02/2014

11. Hanna, G. (2000). "Proof, Explanation and Exploration: An overview." *Educational Studies in Mathematics* Vol. **44**: pp 5-23.

12. Harel, G., Sowder, L. (1998). "Students' Proof Schemes: results from Exploratory Studies." *CBMS Issues in Mathematics Education.* Vol. **7**.

13. Healy, L., Hoyles, C. (2000). "A Study of proof Conceptions in Algebra." *Journal for Research in Mathematics Education.* Vol. **31**, No.4, pp396-428.

14. Hersh, R. (1993). "Proving is Convincing and Explaining." *Educational Studies in Mathematics.* Vol **24**: pp389-399.

15. *Honey & Mumford Learning styles.*

    `http://www2.le.ac.uk/departments/gradschool/training/`

    `eresources/teaching/theories/honey-mumford.`

    Accessed 20/02/2014. Based on: Honey, P. & Mumford, A. (1982). *Manual of Learning Styles.* London.

16. Hoyles, C., Jones, K. (1998), "Proof in Dynamic Geometry Contexts." *In*: Mammana, C., & Villani, V. (Eds)., *Perspectives on the Teaching of Geometry for the 21st Century.* Dordrecht: Kluwer. pp121-128.

17. *Introduction to Research and Research Methods.* Bradford School of Management.
`http://www.brad.ac.uk/management/media/management/els/`
`Introduction-to-Research-and-Research-Methods.pdf`
Accessed 20/02/2014.

18. Jarvis, P. (1987). *Adult Learning in the Social Context.* Croom Helm, New York.

19. Jones, K. (1997). "Student teachers' conceptions of mathematical proof." *Mathematics Education Review*, Vol. **9**, pp21-32.

20. Jones, K. (2000). "The Student Experience of Mathematical Proof at University Level." *International Journal of Mathematics Education, Science and Technology.* Vol. **31**, No. 1. pp53-60.

21. Knuth, E.J. (2002). "Teacher's Conceptions of Proof in the Context of Secondary School Mathematics." *Journal of Mathematics Teacher Education.* Vol. **5**: pp61–88.

22. Kolb, D.A. (1984). *Experiential learning: experience as the source of learning and development.* Prentice-Hall Inc, Englewood Cliffs, New Jersey.

23. Kolb, A.Y. & Kolb, D.A. (2005). *The Kolb Learning Style Inventory—Version 3.1. 2005 Technical Specifications.* Experience Based Learning Systems, Inc & Case Western Reserve University. 15th May 15, 2005.

24. Kolb, D.A. (1981). *Learning Styles and Disciplinary Differences.* Jossey-Bass Limited. 28 Banner Street, London.

25. Leikin, R., Grossman, D. (2013). "Teachers Modify Geometry Problems: from Proof to Investigation." *Educ. Stud. Math.* Vol. **82**. pp 515–531

26. MacDonald, S., Headlam, N. (2009). *Research Methods Handbook.* Centre for Local Economic Strategies (CLES). `http://www.cles.org.uk/publications/research-methods-handbook/`

27. Mainemelis, C., Boyatziz, R., Kolb, D.A. (2000). "Experiential Learning Theory: Previous Research and New Directions." *Perspectives on cognitive, learning, and thinking styles.* Lawrence Erlbaum, New Jersey.

28. McLeod, S. A. (2010). Kolb - Learning Styles. *Simply Psychology* online article
`http://www.simplypsychology.org/learning-kolb.html`
Accessed 29/1/14.

29. Nyaumwe, L., Buzuzi, G. (2007). "Teacher's Attitudes Towards Proof of Mathematical Results in the Secondary School Curriculum: the Case of Zimbabwe." *Mathematics Education Research Journal.* Vol. **19**, No. 3, 21–32.

30. *Oxford Dictionary of English* (2005). 2nd Edition (Revised). Oxford University Press.

31. Polya, G. (1957). *How To Solve It.* 2nd Edition. Princeton University Press, 1957, ISBN 0-691-08097-6. Summary at `http://math.berkeley.edu/~gmelvin/polya.pdf`. Accessed 18/12/13.

32. Rajasekar, S., Philominathan, P., Chinnathambi. V. (2013). "Research Methodology."
`arXiv:physics/0601009v3 [physics.ed-ph]`.

33. Research in Education (RESINED). Plymouth University, UK.
`http://www.edu.plymouth.ac.uk/resined/resedhme.htm`. Accesed 20/2/2014.

34. *Research Methodology - An introduction.*
`http://www.limat.org/data/research/Research Methodology.pdf`. Accessed 20/02/2014.

35. Stylianides, G., Stylianides, A.J. (2006). "Making Proof Central to Pre-High School Mathematics is an Appropriate Instructional Goal: Provable, Refutable, or Undecidable Proposition?" *Proc. 30*$^{\text{th}}$ *Conference of the International Group for the Psychology of Mathematics Education.* Vol. **5**, pp. 209-216.

36. Svinicki, M.D., Dixon, N.M. (1987). "The Kolb Model Modified for Classroom Activities." College Teaching. Vol **35** (4). pp141-146.

37. Tall, D. (1994). "Understanding the Process of Advanced Mathematical Thinking.*" Transcript of an invited ICMI lecture at the International Congress of Mathematicians, Zurich.*

38. Thurston, W.P. (1994). On Proof and Progress in Mathematics. *Bulletin of the American Mathematical Society.* Vol. **30**, No. 2, April 1994, pp 161-177.

39. Vygotsky, L.S.,1978. *Mind and society: The development of higher mental processes.* Cambridge, MA: Harvard University Press.

40. Weber, K., Brophy, A. & Lin, K. (2007). "Learning Advanced Mathematical Concepts by Reading Text." *Proceedings of Research in Undergraduate Mathematics Education*, SanDiego, CA.

41. Woollard, J. *Eduactional Research Methods.* `http://www.pgce.soton.ac.uk/edreme/`.
Accessed 20/02/2014.

6 Glossary

The following mathematically-related definitions are taken from the *Oxford English Dictionary* (2005).

Conjecture

1. An opinion or conclusion formed on the basis of incomplete information.
2. An unproven mathematical or scientific theorem.

Origin: Late Middle English: (in the sense ‘to divide’ and ’divination’): from Old French, or from Latin *conjectura*, from *conicere* ‘put together in thought’, from *con* -‘together’ + *jacere* ‘throw.’

Epistemology

1. Theory of knowledge, especially with regards to its methods, validity and scope, and the distinction between justified belief and opinion.

Origin: Mid 19^{th} century: from Greek *episteme* ‘knowledge’, from *epistasthai* ‘know, know how to do’.

Heuristics

1. Enabling a person to discover or learn something for themselves.
2. Proceeding to a solution by trial and error or by rules that are only loosely defined.

Origin: Early 19^{th} century: formed irregularly from Greek *heuriskein* ‘find’.

Lemma

1. A subsidiary or intermediate theorem in a argument or proof.
2. A heading indicating the subject or argument of a literary composition or annotation.

Origin: Late 16^{th} century: via Latin from Greek *lemma* ‘something assumed’; related to *lambanein* ‘take’.

Proof

1. Evidence or argument establishing a fact or the truth of a statement.
2. A series of stages in the resolution of a mathematical problem.

Origin: Latin *probare* ‘to test’; Late latin *proba*; Old French *proeve*; Middle English *preve.*

Theorem

1. A general proposition not self-evident but proved by a chain of reasoning; a truth established by means of accepted truths.
2. A rule in algebra or other branches of Mathematics expressed by symbols or formulae.

Origin: Mid 16th century: from French *théorème*, or via late Latin from Greek *theōrema* ‘speculation, proposition’, from *theōrein* ‘look at’, from *theōros* ‘spectator.’

7 Appendix 1: Circle Theorems Resources

In this section I present the resources I have created as part of this intervention study. In addition I have also made use of 'Inners' which are an assemblage of questions from textbooks that I give to the boys during classes to help consolidate concepts and tease out any misconceptions at an early stage. Since the completion of the project, Circle Theorem questions have been a regular feature in the 'Outers' (homework) the boys receive. Classes A and B both get two sets of thirty minutes each week. Each of these typically takes the form of a double sided A4 sheet and comprises around 5 questions on a variety of topics (i.e. not just the ones we are studying in class at present).

Resources are:

1. Circle Theorems Proof test
2. Circle Theorems Problems test (questions are adapted from textbook resources questions, but solutions are mine)
3. Circle Theorems handout
4. Lines and angles handout
5. Polygons handout
6. Legacy handwritten Circle Theorems handout

8 Appendix 2: Statistical Analysis of Circle Theorem tests

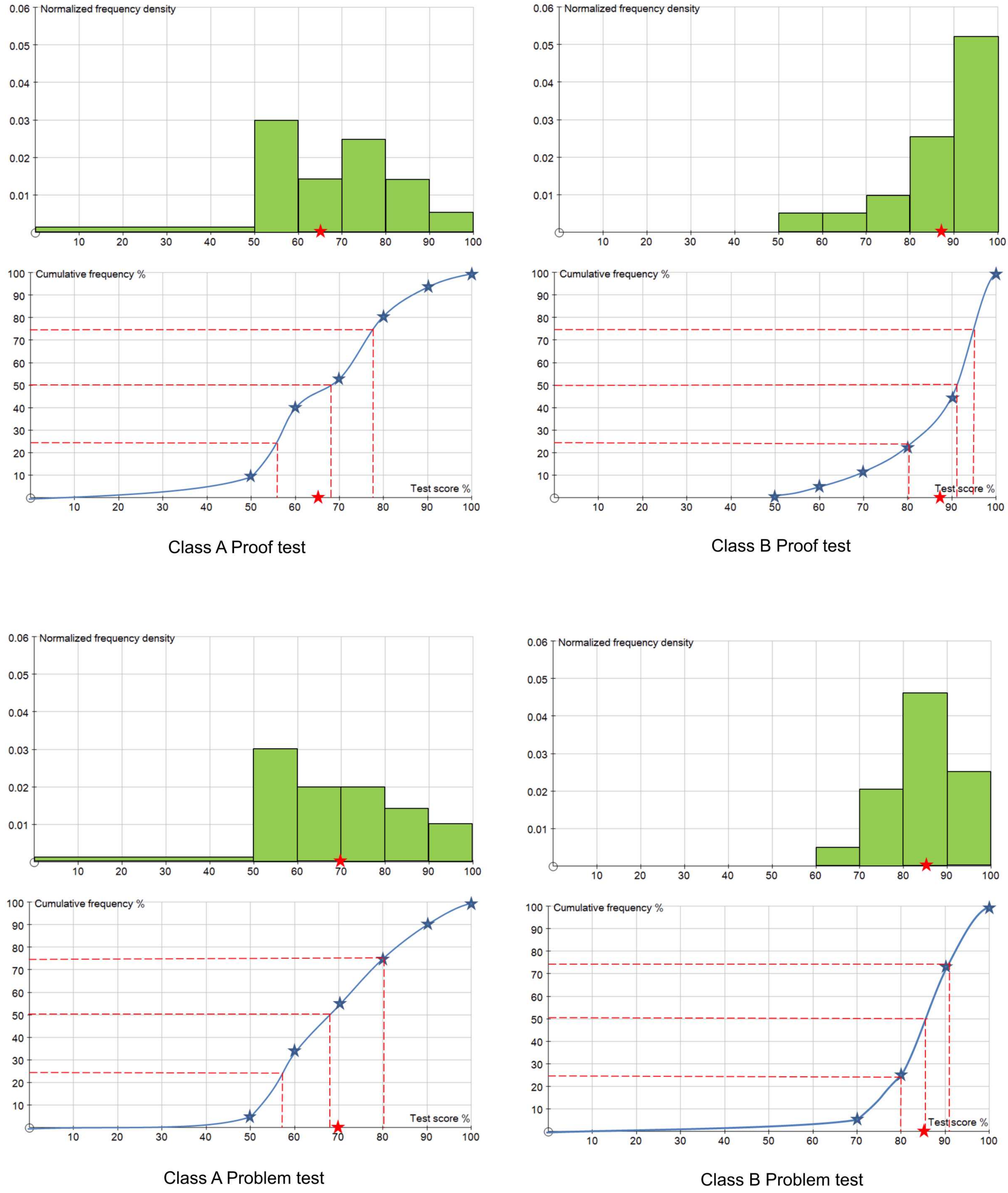


Figure 6: Statistical analysis of test results for Circle Theorem Proof and Problem tests. The same tests were given to classes A and B. As expected, class B (which is a higher ability set in the year above class A) scored significantly higher. Although the distribution of marks for class A implies most students have a sound introductory understanding of the topic, histogram peaks at the lower mark range would suggest significant revision and consolidation is required for pupils in this class (which is indeed what happens at our school both at the week-by-week level and over macro year cycles. i.e. the students will meet Circle Theorems again just like class B this year).

9 Appendix 3: Intervention record

This section includes annotated extracts from my Reflective Diary. (I have kept this for the duration of the Schools Direct course). Key events in the intervention are described in chronological order.

1. *Wednesday 8th January 2014*

   Preparatory work for class A. Started with an introduction to Geometry, mixing definitions of angle types and proof of opposite cross angles, Z-angle theorem and finally why sum of interior angles of a triangle are 180^o. This was done in a traditional fashion as an active discussion with me writing on the board and the boys making notes. A brief introduction to the idea of proof, and the difference between *deducing* a mathematical result is true based upon "things that we all agree to be true", (i.e. axioms) and scientific inference, i.e. a hypothesis is *probable* based upon the statistics of unbiased quantitative observation. (I didn't exactly use these words!) Boys then did about twenty simple geometry questions, really absorbed. About 90% correct via show of hands at the end of the lesson.

2. *Thursday 9th January 2014*

   Class A makes further progress with Geometry. Everyone keen and seemed to be getting on with problems set. Proof of sum of angles in a triangle retained with a little prompt. Talked about polygons (regular and otherwise) and sum of exterior and interior angles. Boys very engaged and capable. One boy did loads of extra problems outside of class, in addition to his set homework!

3. *Saturday 11th and Monday 13th January 2014*

   Consolidation of geometry topic and brief detour to idea of reflection and order of rotational symmetry with class A.

4. *Monday 13th January 2014*

   The big proof introduction for class B. Essentially a discussion lesson, culminating in Circle Theorem proofs, but boys pretty much rapt! Much more intensive discussion addressing "what is proof" than with class A. Linked to the scientific method i.e. theory, experiment loop. Difference between inference and proof.

5. *Tuesday 14th January 2014*

   Circle Theorems and proof for class A Boys super engaged. Small behaviour chat also worked very well as boys were a tad chatty at the beginning of the hour. Led boys through the proofs, but the key steps came from them via questioning. How do we convert the Arrowhead into something we know about (i.e. split into triangles)? Then write down some algebra based on what we know. Are there any isosceles triangles? What is the final pattern that emerges?

6. *Friday 17th January 2014*

   An observed hour for class B based on Circle Theorem proofs. Essentially a managed discussion. All main theorems done nice and slowly, including triangle and Z angle! Even managed to fit in about eight minutes of worksheet questions. Very complimentary feedback. Action to raise the issue of the teaching of proof with the rest of the department.

7. *Saturday 18th January 2014*

   Alternate Segment Theorem and miscellaneous problems workshop with class B.

   Simpler problems and recap of proofs of *Arrowhead Theorem* etc for class A.

8. *Monday 20th January 2014*

   Discussion of 'what is proof?' essay question (set for homework and now marked) with class B.

9. *Tuesday 21st January 2014*

   A nice hard problem involving pretty much all the Circle Theorems. Talked through this slowly on the board, then workshopped the rest of the hour with boys doing problems. In the last five minutes I used my head of department's suggestion of a circle tangent 'approached dynamically' from an extension of the chord in a right angled diameter construction. This seemed to go down well. The key feature is to extend the chord beyond the circle by a fixed amount in every case. This clearly becomes the tangent as the moveable point on the circle approaches the diameter line.

10. *Wednesday 22nd January 2014*

    Circle Theorems Proof test with classes A and B. Same test.

11. *Thursday 23rd January 2014*

    Circle Theorems Problems test with classes A and B.

12. *Monday 27th January 2014*

    Class A: Washup of proof test and recap of *Cyclic Quadrilateral* and *Alternate Segment Theorems* in the context of the others. Boys pretty engaged. Essentially a group discussion.

    Proof ideas were discussed in the Mathematics departmental meeting. I suggested algebraic proofs related to the addition and multiplication of odd and even integers as well as geometrical results. This was very well received.

13. *Tuesday 28th January*

    Continued discussion of tests of Circle Theorems with class A.

    Feedback of tests for class B interleaved with other topics. Results for both classes very positive. (See Analysis).

14. *Wednesday 5th March 2014*

    Construction of the regular pentagon presentation delivered to the department at our weekly meeting. I decided to get them all to actually do the construction first, using compasses and rulers! Presentation went down well and lots of fruitful discussion ensued. Lovely construction from the 'elder statesman of the department' completed the piece, which directly proved $\cos 36^{o}$ = half the golden ratio i.e. $\frac{1}{2} \times \frac{1+\sqrt{5}}{2}$

15. Note: The deputy head of department appears to be suitably inspired by my proof intervention that he is creating a set of stretching Circle Theorem problems. This has a good chance of being completed by next year and should be an excellent resource for the school.