Notes on the collision of two masses

Dr Andrew French
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1 Summary

- In this monograph we derive an equation for the post impact velocities of two masses with known pre-impact velocities \( u_1 \) and \( u_2 \) and masses \( m_1 \) and \( m_2 \). This includes a coefficient of restitution \( (C) \) which models the continuum between elastic \( (C = 1) \) and inelastic \( (C = 0) \) collisions.
- Computes the loss \( \Delta E \) of kinetic energy during a collision.
- Introduces the concept of the Zero Momentum Frame. This is an example of the use of reference frame transformation and also, interestingly, a description of the law of conservation of momentum using a symmetry argument.
- The elastic special case is extended to investigate the limit \( m_2 \gg m_1 \). This yields the marvellous result that \( m_1 \) will rise to a maximum of nine times the original height, if both masses are dropped together in a uniform gravitational field (i.e. typical of a secondary school physics laboratory!), with the smaller mass uppermost.
- Uses vector notation throughout, keeping results general for any coordinate system.

2 Assumptions

- Two ‘point masses’. i.e. purely linear motion, no rotational modes.
- Impact of external forces (e.g. gravity, friction) negligible on timescales of collision.
- The masses actually approach each other. We can therefore choose a frame of reference whereby their momenta are equal in magnitude but opposite in direction. (The Zero Momentum Frame - see below).
- Classical dynamics, speeds \( \ll c \) and therefore no relativistic effects. Therefore Galilean transforms and Newtonian dynamics.
- Micro nature of collision (i.e. degree of elasticity or inelasticity) is wrapped up in the empirical coefficient of restitution.

Figure 1: Collision of masses, prior to impact.

Figure 2: Collision of two masses, post impact.
3 Define the Zero Momentum Frame

Transform generic frame into a Zero Momentum Frame (ZMF), i.e. subtract \( V \) from all velocity vectors such that total momentum \( P_{\text{total}} \) in this frame is zero.

\[
P_{\text{total}} = \sum_i m_i (u_i - V) = 0
\]  

(1)

Since there are only two masses in our system

\[
\sum_i m_i (u_i - V) = m_1 (u_1 - V) + m_2 (u_2 - V) = 0
\]  

(2)

Therefore

\[
V = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}
\]  

(3)

4 Coefficient of Restitution

Let \( u_1 \) and \( u_2 \) be the velocity vectors of the masses prior to impact and define \( v_1 \) and \( v_2 \) to be the corresponding vectors after impact. If the collision is perfectly elastic then the masses will part with relative speed \( |v_2 - v_1| \) equal to that of approach \( |u_2 - u_1| \). If the collision is not perfectly elastic then let us generalize by defining a coefficient of restitution \( C \) which defines the ratio of parting speed to approach speed. Note \( C \) takes the same form whether the system is viewed in the ZMF, or any other constant velocity frame.

\[
C = \frac{|v_2 - V - (v_1 - V)|}{|u_2 - V - (u_1 - V)|} = \frac{|v_2 - v_1|}{|u_2 - u_1|}\]

(4)

\( C = 1 \) implies a fully elastic collision,\(^1\) whereas \( C = 0 \) implies the masses stick together following collision. This is called a fully inelastic collision.

5 Use the ZMF to compute the result of collision

As shown in Figure 3, the ZMF allows us to predict the AFTER IMPACT situation using a symmetry argument. If two objects collide with momenta of equal magnitude but opposing direction, the result will be a reversal of the directions of momenta. The coefficient of restitution \( C \) sets the magnitude of the reversal.

\(^1\)It might be possible that \( C > 1 \) if, following collision, extra energy (stored within each of the two masses) is released. For example, two explosive shells colliding.

Figure 3: In the Zero Momentum Frame, *symmetry* is used to predict the AFTER IMPACT result. i.e. a reversal of velocity vectors, scaled by the coefficient of restitution $C$. 

$C$ is the coefficient of restitution
Figure 4: Post collision velocities are computed in terms of pre-impact velocities \( u_1 \) and \( u_2 \), coefficient of restitution \( C \) and Zero Momentum Frame velocity \( V \).

The post collision situation in the generic frame is computed by adding \( V \) to resultant velocities following collision in the ZMF.

\[
\begin{align*}
v_1 &= C(V - u_1) + V \\ v_2 &= C(V - u_2) + V
\end{align*}
\]  

which simplifies to

\[
\begin{align*}
v_1 &= (1 + C)V - Cu_1 \\ v_2 &= (1 + C)V - Cu_2
\end{align*}
\]
Substituting for $V = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}$ gives the following expressions for the impact velocities in terms of the initial knowns: $u_1, u_2, m_1, m_2, C$.

\begin{align*}
  v_1 &= u_1 \left\{ \frac{m_1 (1 + C)}{m_1 + m_2} - C \right\} + u_2 \frac{m_2 (1 + C)}{m_1 + m_2} \\
  v_2 &= u_1 \frac{m_1 (1 + C)}{m_1 + m_2} + u_2 \left\{ \frac{m_2 (1 + C)}{m_1 + m_2} - C \right\}
\end{align*}

which simplifies to

\begin{align*}
  v_1 &= u_1 \left\{ \frac{m_1 - C m_2}{m_1 + m_2} \right\} + u_2 \frac{m_2 (1 + C)}{m_1 + m_2} \\
  v_2 &= u_1 \frac{m_1 (1 + C)}{m_1 + m_2} + u_2 \left\{ \frac{m_2 - C m_1}{m_1 + m_2} \right\}
\end{align*}

The kinetic energy pre-impact is

$$T_{pre} = \frac{1}{2} m_1 |u_1|^2 + \frac{1}{2} m_2 |u_2|^2$$

The kinetic energy post-impact is

\begin{align*}
  T_{post} &= \frac{1}{2} m_1 |v_1|^2 + \frac{1}{2} m_2 |v_2|^2 \\
  &= \frac{1}{2} \frac{m_1}{(m_1 + m_2)^2} |u_1 (m_1 - C m_2) + u_2 m_2 (1 + C)|^2 \\
  &\quad + \frac{1}{2} \frac{m_2}{(m_1 + m_2)^2} |u_1 m_1 (1 + C) + u_2 (m_2 - C m_1)|^2 \\
  &= \frac{m_1 m_2}{2 (m_1 + m_2)} \left\{ A |u_1|^2 + B |u_2|^2 + 2 u_1 \cdot u_2 (1 - C^2) \right\}
\end{align*}

where

\begin{align*}
  A &= \frac{(m_1 - C m_2)^2 + m_1 m_2 (1 + C)^2}{(m_1 + m_2) m_2} \\
  B &= \frac{(m_2 - C m_1)^2 + m_1 m_2 (1 + C)^2}{(m_1 + m_2) m_1}
\end{align*}

Note the last step requires a few additional lines of algebra.
6 Special cases

Consider two special cases:

6.1 Special case #1: Inelastic collision. i.e. \( C = 0 \)

Setting \( C = 0 \) in Equation 8:

\[
\begin{align*}
\mathbf{v}_1 &= \mathbf{u}_1 \frac{m_1}{m_1 + m_2} + \mathbf{u}_2 \frac{m_2}{m_1 + m_2} \\
\mathbf{v}_2 &= \mathbf{u}_1 \frac{m_1}{m_1 + m_2} + \mathbf{u}_2 \frac{m_2}{m_1 + m_2}
\end{align*}
\]

i.e. \( \mathbf{v}_1 = \mathbf{v}_2 \) as expected. Masses move together as one with the same impact velocity. The post collision kinetic energy is given by

\[
T_{post} = \frac{1}{2} \left( \frac{m_1}{m_1 + m_2} \right) \mathbf{u}_1 \left( \frac{m_1}{m_1 + m_2} \mathbf{u}_1 + \frac{m_2}{m_1 + m_2} \mathbf{u}_2 \right)^2
\]

\[
= \frac{1}{2} \frac{m_1^2 |\mathbf{u}_1|^2 + 2m_1 m_2 |\mathbf{u}_2|^2 + 2m_1 m_2 \mathbf{u}_1 \cdot \mathbf{u}_2}{m_1 + m_2}
\]

The kinetic energy loss (most likely converted into heat or the work done in deforming the masses as they stick) as a result of the inelastic collision process is therefore

\[
\Delta E = T_{pre} - T_{post} = \frac{1}{2} m_1 |\mathbf{u}_1|^2 + \frac{1}{2} m_2 |\mathbf{u}_2|^2 - \frac{1}{2} \frac{m_1^2 |\mathbf{u}_1|^2 + m_2^2 |\mathbf{u}_2|^2 + 2m_1 m_2 \mathbf{u}_1 \cdot \mathbf{u}_2}{m_1 + m_2}
\]

\[
= \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} \left( |\mathbf{u}_1|^2 + |\mathbf{u}_2|^2 - 2 \mathbf{u}_1 \cdot \mathbf{u}_2 \right)
\]

\[
= \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} |\mathbf{u}_1 - \mathbf{u}_2|^2
\]

\[
= \frac{1}{2} m_1 |\mathbf{u}_1 - \mathbf{u}_2|^2
\]

This makes sense in the limit \( m_2 \gg m_1 \). Imagine throwing a 100g ball of plasticine on the ground. \( m_2 \) is the Earth at \( 5.97 \times 10^{24} \) kg. If we choose a reference frame fixed to the ground \( \mathbf{u}_2 = 0 \). The loss in kinetic energy is therefore \( \Delta E = \frac{1}{2} m_1 |\mathbf{u}_1|^2 \). i.e. the entire pre-collision amount associated with the plasticine.
6.2 Special case #2: Elastic collision. i.e. $C = 1$

Setting $C = 1$ in Equation 8:

\[
v_1 = u_1 \left\{ \frac{2m_1}{m_1 + m_2} - 1 \right\} + u_2 \frac{2m_2}{m_1 + m_2} + \frac{2m_2}{m_1 + m_2} - 1
\]

The total kinetic energy post-impact is:

\[
T_{\text{post}} = \frac{1}{2}m_1 |v_1|^2 + \frac{1}{2}m_2 |v_2|^2
\]

\[
= \frac{1}{2}m_1 \left| u_1 \left\{ \frac{2m_1}{m_1 + m_2} - 1 \right\} + u_2 \frac{2m_2}{m_1 + m_2} \right|^2 + \frac{1}{2}m_2 \left| u_1 \frac{2m_1}{m_1 + m_2} + u_2 \left\{ \frac{2m_2}{m_1 + m_2} - 1 \right\} \right|^2
\]

\[
= \frac{1}{2}m_1 \left| u_1 \left\{ \frac{m_1 - m_2}{m_1 + m_2} \right\} + u_2 \frac{2m_2}{m_1 + m_2} \right|^2 + \frac{1}{2}m_2 \left| u_1 \frac{2m_1}{m_1 + m_2} + u_2 \left\{ \frac{m_2 - m_1}{m_1 + m_2} \right\} \right|^2
\]

\[
= \frac{1}{2}m_1 \left| u_1 \right|^2 \left\{ \frac{m_1 - m_2}{m_1 + m_2} \right\}^2 + \left| u_2 \right|^2 \frac{4m_2^2}{(m_1 + m_2)^2} + 4u_1 \cdot u_2 \left\{ \frac{m_1 - m_2}{m_1 + m_2} \right\} \frac{m_2}{m_1 + m_2}
\]

\[
+ \frac{1}{2}m_2 \left| u_2 \right|^2 \left\{ \frac{m_1 - m_2}{m_1 + m_2} \right\}^2 + \left| u_1 \right|^2 \frac{4m_1^2}{(m_1 + m_2)^2} - 4u_1 \cdot u_2 \left\{ \frac{m_1 - m_2}{m_1 + m_2} \right\} \frac{m_1}{m_1 + m_2}
\]

\[
= \frac{1}{2}m_1 \left| u_1 \right|^2 \left\{ \frac{m_1 - m_2}{m_1 + m_2} \right\}^2 + \frac{4m_1 m_2}{(m_1 + m_2)^2} + \frac{1}{2}m_2 \left| u_2 \right|^2 \left( \frac{4m_1 m_2}{(m_1 + m_2)^2} + \left\{ \frac{m_1 - m_2}{m_1 + m_2} \right\}^2 \right)
\]

\[
= \frac{1}{2}m_1 \left| u_1 \right|^2 + \frac{1}{2}m_2 \left| u_2 \right|^2
\]

Since \( \left\{ \frac{m_1 - m_2}{m_1 + m_2} \right\}^2 + \frac{4m_1 m_2}{(m_1 + m_2)^2} = \frac{m_1^2 + m_2^2 - 2m_1 m_2 + 4m_1 m_2}{(m_1 + m_2)^2} = \frac{m_1^2 + m_2^2 + 2m_1 m_2}{(m_1 + m_2)^2} = \frac{(m_1 + m_2)^2}{(m_1 + m_2)^2} = 1 \)

Hence in an elastic collision, kinetic energy is conserved. i.e. $\Delta E = T_{\text{pre}} - T_{\text{post}} = 0$
6.3 An interesting result!

Consider an elastic collision where $m_2 \gg m_1$. Using 19:

$$v_1 = 2u_2 - u_1 \quad \text{(21)}$$
$$v_2 = u_2$$

If masses are dropped together from height $h$ in a uniform gravitational field (with field strength $g$), and the larger mass strikes the ground elastically before meeting the smaller mass

$$u_1 = -u\hat{x} \quad \text{(22)}$$
$$u_2 = u\hat{x}$$

Where $\hat{x}$ is a unit vector in the ‘up’ direction.

Therefore if $v_1 = v_1\hat{x}$ and $v_2 = v_2\hat{x}$

$$v_1 = 3u \quad \text{(23)}$$
$$v_2 = u$$

By conservation of energy we can calculate the height risen by both masses following collision. i.e. the height at which the velocity of each mass becomes zero

$$m_1gH = \frac{1}{2}mv_1^2 \quad \text{(24)}$$
$$m_2gh = \frac{1}{2}m_2u^2 = \frac{1}{2}m_2v_2^2$$

Hence $H = \frac{9u^2}{2g}$ and $h = \frac{u^2}{2g}$, which yields the fascinating result

$$H = 9h \quad \text{(25)}$$

Figure 5: Elastic balls of significantly different mass (e.g. a basket ball and a ping pong ball) are dropped together from height $h$.

Figure 6: An interesting result! In the limit $m_2 \gg m_1$, two balls dropped together (with the least massive uppermost) will cause the lighter mass to rise up to nine times the original height, if all collisions are perfectly elastic.

Figure 7: Recoil velocity of mass #1 plotted against coefficient of restitution $C$ and mass ratio $\frac{m_2}{m_1}$. Approach speeds of both masses are $u$. Note for elastic collisions in the limit $m_2 \gg m_1$ the recoil velocity tends to $3u$.

7 Graphical exploration

The variation of $v_1, v_2, \frac{T_{post} - T_{pre}}{T_{pre}}$ with $C$ and $\frac{m_2}{m_1}$ can be explored graphically using plotting software such as MATLAB and the following simplifications:

$$u_1 = -\alpha u \hat{x}$$  (26)
$$u_2 = u \hat{x}$$
$$v_1 = v_1 \hat{x}$$  (27)
$$v_2 = v_2 \hat{x}$$  (28)

where $\hat{x}$ is a unit vector and $\alpha$ is a scalar parameter. For the plots in Figures 7 to 9, $\alpha = 1$. The general results in equations 8, 10 and 9 simplify to

$$v_1 = \frac{u}{m_1 + m_2} (-\alpha (m_1 - C m_2) + m_2 (1 + C))$$  (29)
$$v_2 = \frac{u}{m_1 + m_2} (m_2 - C m_1 - \alpha m_1 (1 + C))$$
$$T_{pre} = \frac{1}{2} u^2 (m_1 \alpha^2 + m_2)$$  (30)
$$T_{post} = \frac{m_1 m_2 u^2}{2 (m_1 + m_2)} \left\{ A \alpha^2 + B - 2 \alpha (1 - C^2) \right\}$$  (31)
Figure 8: Recoil velocity of mass #2 plotted against coefficient of restitution $C$ and mass ratio $\frac{m_2}{m_1}$. Approach speeds of both masses are $u$.

Figure 9: Fractional loss of total kinetic energy (KE) of two masses following collision. Both masses approach each other with speed $u$. For an inelastic collision ($C = 0$) the maximum loss (100%) of KE is for $m_1 = m_2$. For elastic ($C = 1$) collisions there is no loss of KE.
8 Extension: ‘The Irish Moon Shot’

The surprising result that the elastic collision of two spheres results in a speed multiplication (of the least massive) by up to three times (in the limit that the heavier sphere is infinitely heavier) can be used to explore the following question:

“$N$ elastic spheres are stacked and dropped as an ensemble from a height $h$ of one metre, reaching a velocity $u = \sqrt{2gh}$. The uppermost sphere has mass $m$ and subsequent spheres increase in mass by a constant factor $k$. If all spheres collide with coefficient of restitution $C$ and the inter-sphere collisions can be modelled separately$^2$, how many spheres are needed to cause the uppermost mass to escape the Earth’s gravitational field?”

Solution

Consider the collision between mass $m_n$ and $m_{n+1}$. If $\hat{x}$ is a unit vector pointing up (i.e. in the opposite direction to the local gravitational field $g$) the pre-collision velocities are:

\[ u_n = -u \hat{x} \]
\[ u_{n+1} = v_{n+1} \hat{x} \]

The post-collision velocity of mass $m_n$ is $v_n = v_n \hat{x}$. The general result in equation 8 states

\[ v_n = u_n \left\{ \frac{m_n - C m_{n+1}}{m_n + m_{n+1}} \right\} + u_{n+1} \frac{m_{n+1}(1 + C)}{m_n + m_{n+1}} \]

Hence:

\[ v_n = \frac{-m_n + C m_{n+1}}{m_n + m_{n+1}} u + v_{n+1} \frac{m_{n+1}(1 + C)}{m_n + m_{n+1}} \]

Now $\frac{m_{n+1}}{m_n} = k$, therefore

\[ v_n = \frac{kC - 1}{1 + k} u + v_{n+1} \frac{k(1 + C)}{1 + k} \]

Which simplifies to

\[ v_{n+1} = \frac{1 + k}{k(1 + C)} v_n - \frac{kC - 1}{k(1 + C)} u \]

Let us consider the first few terms

\[ v_2 = av_1 - b \]
\[ v_3 = av_2 - b = a(av_1 - b) - b = a^2v_1 - ab - b \]
\[ v_4 = av_3 - b = a(a^2v_1 - ab - b) - b = a^3v_1 - a^2b - ab - b \]
\[ v_5 = av_4 - b = a^4v_1 - a^3b - a^2b - ab - b \]

By spotting the geometric pattern we can write (for $n > 1$)

\[ v_n = a^{n-1}v_1 - b \sum_{y=0}^{n-2} a^y \]

This summation can be evaluated using the standard result

\[ S_n = a \sum_{i=0}^{n-1} r^i = a \frac{1 - r^n}{1 - r} \]

$^2$Rather than a multi body collision, which may not have an analytic solution.

Hence

\[ \sum_{y=0}^{n-2} a^y = \frac{1 - a^{n-1}}{1 - a} \]  

which gives

\[ v_n = a^{n-1}v_1 - b \frac{1 - a^{n-1}}{1 - a} \]  

Now mass \( v_N \) collides with the Earth with pre-collision velocity \( u_N = -u \hat{x} \) and therefore rebounds with velocity \( v_N = uC \hat{x} \). Therefore

\[ uC = a^{N-1}v_1 - b \frac{1 - a^{N-1}}{1 - a} \]  

which can be rearranged to yield an expression for the recoil velocity \( v_1 \) of the uppermost mass.

\[ v_1 = \frac{uC}{a^{N-1}} + b \frac{1 - a^{N-1}}{(1 - a)a^{N-1}} \]  

\[ = uCa^{1-N} + b \frac{(a^{1-N} - 1)}{1 - a} \]  

Since \( a = \frac{1+k}{k(1+C)} \) and \( b = \frac{kC-1}{k(1+C)}u \) this gives

\[ \frac{v_1}{u} = C \left( \frac{1+k}{k(1+C)} \right)^{1-N} + kC - 1 \frac{ \left( \frac{1+k}{k(1+C)} \right)^{1-N} - 1}{1 - \frac{1+k}{k(1+C)}} \]  

\[ = C \left( \frac{1+k}{k(1+C)} \right)^{1-N} + kC - 1 \frac{ \left( \frac{1+k}{k(1+C)} \right)^{1-N} - 1}{ \frac{k(1+C)-1-k}{k(1+C)}} \]  

\[ = C \left( \frac{1+k}{k(1+C)} \right)^{1-N} + \left( \frac{1+k}{k(1+C)} \right)^{1-N} - 1 \]  

Hence

\[ \frac{v_1}{u} = \left( \frac{k(1+C)}{1+k} \right)^{N-1} (C + 1) - 1 \]  

Now consider the special case when \( k = 2 \) and \( C = 1 \)

\[ \frac{v_1}{u} = 2 \left( \frac{4}{3} \right)^{N-1} - 1 \]  

Now for mass \( m_1 \) to escape the Earth’s gravitational field

\[ v_1 = \sqrt{\frac{2GM}{R_E}} \]  

where \( G = 6.67 \times 10^{-11} \text{m}^3\text{kg}^{-1}\text{s}^{-2} \), \( M = 5.97 \times 10^{24} \text{kg} \) and \( R_E = 6.38 \times 10^6 \text{m} \). Therefore if \( u = \sqrt{2gh} \) where \( h \) is the initial height dropped (\( h = 1 \text{m} \) in our example)

\[ \sqrt{\frac{GM}{R_Egh}} = 2 \left( \frac{4}{3} \right)^{N-1} - 1 \]
Which implies

\[ N = \text{ceil} \left\{ 1 + \frac{\log \left( \sqrt{\frac{GM}{R_Eg}} + 1 \right) - \log 2}{\log 4^\frac{4}{3}} \right\} = 26 \]

where ‘ceil’ rounds up (25.82) to the nearest integer.

So is this a practical moonshot? If the uppermost mass is 1kg this means the 26th mass is \(2^{25} \approx 33,554\) metric tonnes. This implies some very large elastic balls!

Perhaps more practical example is to consider \(k = 2, N = 4\) and \(C = 1\). This is possibly characteristic of the Irish Moonshot toy available from good science retailers. In this case \(\frac{\text{v}_u}{v} = 2 \left(\frac{4}{3}\right)^3 - 1 = \frac{128}{27} - 1 \approx 3.7\).