# Notes on the collision of two masses

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## 1 Summary


- In this monograph we derive an equation for the post impact velocities of two masses with known pre-impact velocities $\mathbf{u}_1$ and $\mathbf{u}_2$ and masses $m_1$ and $m_2$. This includes a *coefficient of restitution* ($C$) which models the continuum between elastic ($C = 1$) and inelastic ($C = 0$) collisions.

- Computes the loss $\Delta E$ of kinetic energy during a collision.

- Introduces the concept of the Zero Momentum Frame. This is an example of the use of reference frame transformation and also, interestingly, a description of the law of conservation of momentum using a *symmetry* argument.

- The elastic special case is extended to investigate the limit $m_2 \gg m_1$. This yields the marvellous result that $m_1$ will rise to a maximum of *nine times* the original height, if both masses are dropped together in a uniform gravitational field (i.e. typical of a secondary school physics laboratory!), with the smaller mass uppermost.

- Uses vector notation throughout, keeping results general for any coordinate system.


## 2 Assumptions

- Two 'point masses'. i.e. purely linear motion, no rotational modes.

- Impact of external forces (e.g. gravity, friction) negligible on timescales of collision.

- The masses actually approach each other. We can therefore choose a frame of reference whereby their momenta are equal in magnitude but opposite in direction. (The Zero Momentum Frame - see below).

- Classical dynamics, speeds $\ll c$ and therefore no relativistic effects. Therefore Galilean transforms and Newtonian dynamics.

- Micro nature of collision (i.e. degree of elasticity or inelasticity) is wrapped up in the empirical coefficient of restitution.

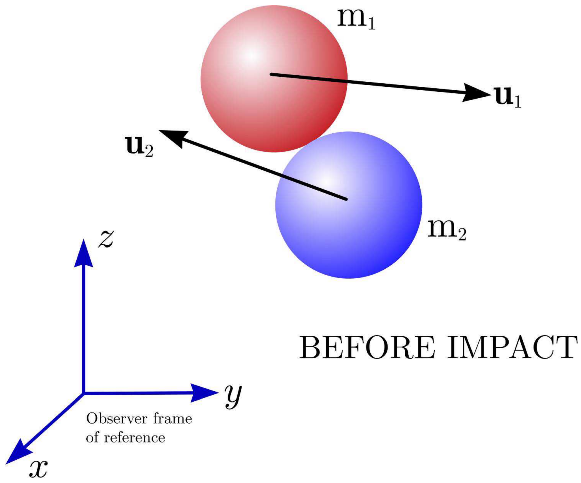


Figure 1: Collision of masses, prior to impact.

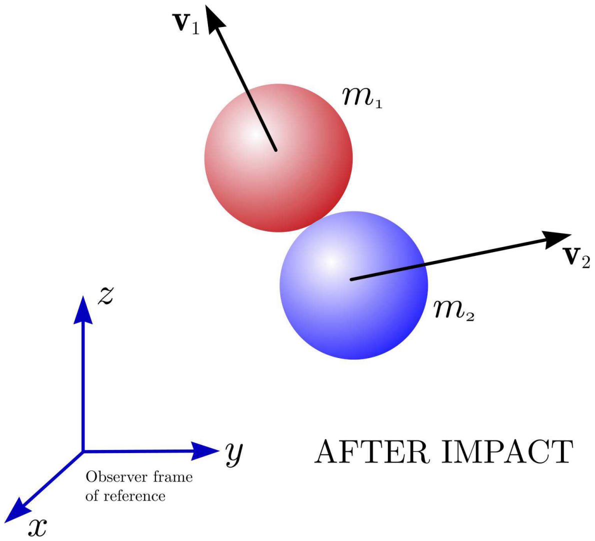


Figure 2: Collision of two masses, post impact.

## 3 Define the Zero Momentum Frame

Transform generic frame into a *Zero Momentum Frame* (ZMF). i.e. subtract $\mathbf{V}$ from all velocity vectors such that total momentum $\mathbf{P}_{total}$ in this frame is zero.

$$\mathbf{P}_{total} = \sum_i m_i(\mathbf{u}_i - \mathbf{V}) = \mathbf{0} \tag{1}$$

Since there are only two masses in our system

$$\sum_i m_i(\mathbf{u}_i - \mathbf{V}) = m_1(\mathbf{u}_1 - \mathbf{V}) + m_2(\mathbf{u}_2 - \mathbf{V}) = \mathbf{0} \tag{2}$$

Therefore

$$\mathbf{V} = \frac{m_1\mathbf{u}_1 + m_2\mathbf{u}_2}{m_1 + m_2} \tag{3}$$

## 4 Coefficient of Restitution

Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be the velocity vectors of the masses prior to impact and define $\mathbf{v}_1$ and $\mathbf{v}_2$ to be the corresponding vectors after impact. If the collision is perfectly *elastic* then the masses will part with relative speed $|\mathbf{v}_2 - \mathbf{v}_1|$ equal to that of approach $|\mathbf{u}_2 - \mathbf{u}_1|$. If the collision is not perfectly elastic then let us generalize by defining a coefficient of restitution $C$ which defines the ratio of parting speed to approach speed. Note $C$ takes the same form whether the system is viewed in the ZMF, or any other constant velocity frame.

$$C = \frac{|\mathbf{v}_2 - \mathbf{V} - (\mathbf{v}_1 - \mathbf{V})|}{|\mathbf{u}_2 - \mathbf{V} - (\mathbf{u}_1 - \mathbf{V})|} = \frac{|\mathbf{v}_2 - \mathbf{v}_1|}{|\mathbf{u}_2 - \mathbf{u}_1|} \tag{4}$$

$C = 1$ implies a fully elastic collision,[1] whereas $C = 0$ implies the masses stick together following collision. This is called a fully *inelastic* collision.

## 5 Use the ZMF to compute the result of collision

As shown in Figure 3, the ZMF allows us to predict the AFTER IMPACT situation using a *symmetry* argument. If two objects collide with momenta of *equal magnitude* but *opposing direction*, the result will be a *reversal of the directions of momenta*. The coefficient of restitution $C$ sets the magnitude of the reversal.

[1] It might be possible that $C > 1$ if, following collision, extra energy (stored within each of the two masses) is released. For example, two explosive shells colliding.

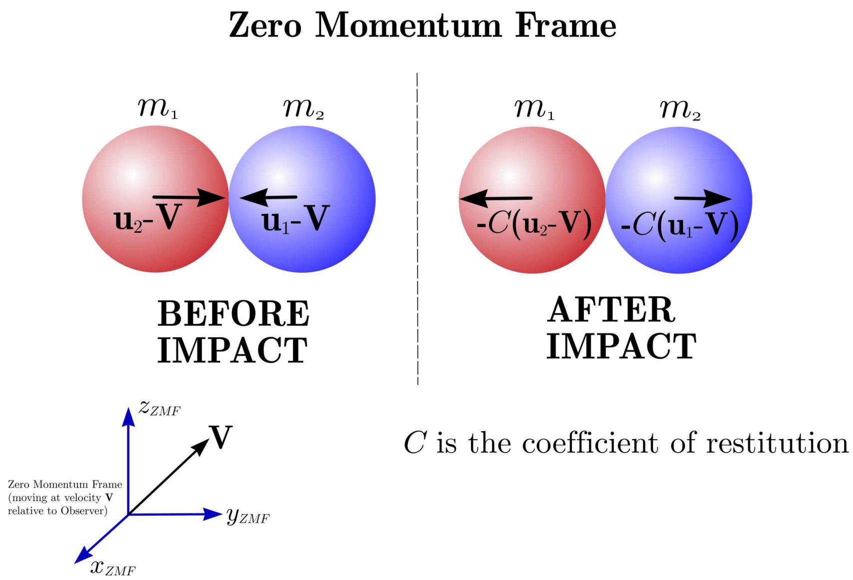


Figure 3: In the Zero Momentum Frame, *symmetry* is used to predict the AFTER IMPACT result. i.e. a reversal of velocity vectors, scaled by the coefficient of restitution $C$.

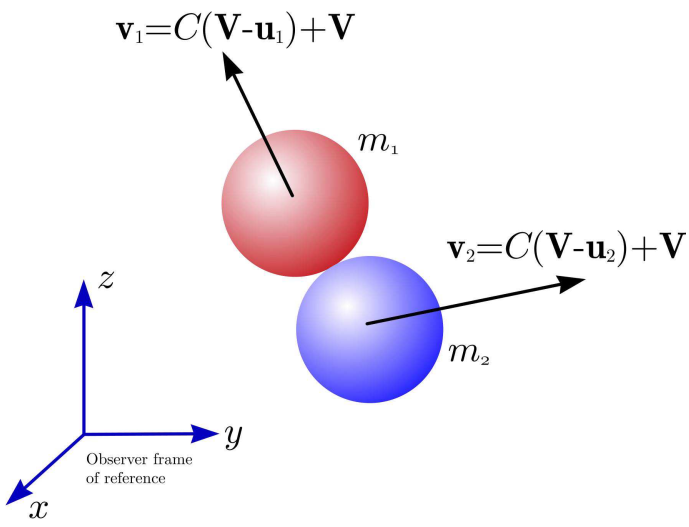


Figure 4: Post collision velocities are computed in terms of pre-impact velocities $\mathbf{u}_1$ and $\mathbf{u}_2$, coefficient of restitution $C$ and Zero Momentum Frame velocity $\mathbf{V}$.

The post collision situation in the generic frame is computed by adding $\mathbf{V}$ to resultant velocities following collision in the ZMF.

$$\begin{aligned} \mathbf{v}_1 &= C(\mathbf{V} - \mathbf{u}_1) + \mathbf{V} \\ \mathbf{v}_2 &= C(\mathbf{V} - \mathbf{u}_2) + \mathbf{V} \end{aligned} \tag{5}$$

which simplifies to

$$\begin{aligned} \mathbf{v}_1 &= (1 + C)\mathbf{V} - C\mathbf{u}_1 \\ \mathbf{v}_2 &= (1 + C)\mathbf{V} - C\mathbf{u}_2 \end{aligned} \tag{6}$$

Substituting for $\mathbf{V} = \frac{m_1\mathbf{u}_1 + m_2\mathbf{u}_2}{m_1+m_2}$ gives the following expressions for the impact velocities in terms of the initial knowns: $\mathbf{u}_1, \mathbf{u}_2, m_1, m_1, C$

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{u}_1 \left\{ \frac{m_1(1+C)}{m_1+m_2} - C \right\} + \mathbf{u}_2 \frac{m_2(1+C)}{m_1+m_2} \\ \mathbf{v}_2 &= \mathbf{u}_1 \frac{m_1(1+C)}{m_1+m_2} + \mathbf{u}_2 \left\{ \frac{m_2(1+C)}{m_1+m_2} - C \right\} \end{aligned} \tag{7}$$

which simplifies to

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{u}_1 \left\{ \frac{m_1 - Cm_2}{m_1+m_2} \right\} + \mathbf{u}_2 \frac{m_2(1+C)}{m_1+m_2} \\ \mathbf{v}_2 &= \mathbf{u}_1 \frac{m_1(1+C)}{m_1+m_2} + \mathbf{u}_2 \left\{ \frac{m_2 - Cm_1}{m_1+m_2} \right\} \end{aligned} \tag{8}$$

The kinetic energy pre-impact is

$$T_{pre} = \frac{1}{2} m_1 |\mathbf{u}_1|^2 + \frac{1}{2} m_2 |\mathbf{u}_2|^2 \tag{9}$$

The kinetic energy post-impact is

$$\begin{aligned} T_{post} &= \frac{1}{2} m_1 |\mathbf{v}_1|^2 + \frac{1}{2} m_2 |\mathbf{v}_2|^2 && (10) \\ &= \frac{1}{2} \frac{m_1}{(m_1+m_2)^2} |\mathbf{u}_1 (m_1 - Cm_2) + \mathbf{u}_2 m_2 (1+C)|^2 \\ &\quad + \frac{1}{2} \frac{m_2}{(m_1+m_2)^2} |\mathbf{u}_1 m_1 (1+C) + \mathbf{u}_2 (m_2 - Cm_1)|^2 \\ &= \frac{m_1 m_2}{2(m_1+m_2)} \left\{ A |\mathbf{u}_1|^2 + B |\mathbf{u}_2|^2 + 2\mathbf{u}_1 \cdot \mathbf{u}_2 (1 - C^2) \right\} && (11) \end{aligned}$$

where

$$\begin{aligned} A &= \frac{(m_1 - Cm_2)^2 + m_1 m_2 (1+C)^2}{(m_1+m_2)\, m_2} \\ B &= \frac{(m_2 - Cm_1)^2 + m_1 m_2 (1+C)^2}{(m_1+m_2)\, m_1} \end{aligned} \tag{12}$$

Note the last step requires a few additional lines of algebra.

# 6 Special cases

Consider two special cases:

## 6.1 Special case #1: Inelastic collision. i.e. $C = 0$

Setting $C = 0$ in Equation 8:

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{u}_1 \frac{m_1}{m_1+m_2} + \mathbf{u}_2 \frac{m_2}{m_1+m_2} \\ \mathbf{v}_2 &= \mathbf{u}_1 \frac{m_1}{m_1+m_2} + \mathbf{u}_2 \frac{m_2}{m_1+m_2} \end{aligned} \tag{13}$$

i.e. $\mathbf{v}_1 = \mathbf{v}_2$ as expected. Masses move together as one with the same impact velocity. The post collision kinetic energy is given by

$$\begin{aligned} T_{post} &= \frac{1}{2}(m_1+m_2)\left|\mathbf{u}_1 \frac{m_1}{m_1+m_2} + \mathbf{u}_2 \frac{m_2}{m_1+m_2}\right|^2 \\ &= \frac{1}{2}\frac{|m_1\mathbf{u}_1 + m_2\mathbf{u}_2|^2}{m_1+m_2} = \frac{1}{2}\frac{m_1^2|\mathbf{u}_1|^2 + m_2^2|\mathbf{u}_2|^2 + 2m_1m_2\mathbf{u}_1\cdot\mathbf{u}_2}{m_1+m_2} \end{aligned} \tag{14}$$

The kinetic energy loss (most likely converted into heat or the work done in deforming the masses as they stick) as a result of the inelastic collision process is therefore

$$\Delta E = T_{pre} - T_{post} = \frac{1}{2}m_1|\mathbf{u}_1|^2 + \frac{1}{2}m_2|\mathbf{u}_2|^2 - \frac{1}{2}\frac{m_1^2|\mathbf{u}_1|^2 + m_2^2|\mathbf{u}_2|^2 + 2m_1m_2\mathbf{u}_1\cdot\mathbf{u}_2}{m_1+m_2} \tag{15}$$

$$= \frac{1}{2}\frac{m_1m_2}{m_1+m_2}\left(|\mathbf{u}_1|^2 + |\mathbf{u}_2|^2 - 2\mathbf{u}_1\cdot\mathbf{u}_2\right) \tag{16}$$

$$= \frac{1}{2}\frac{m_1m_2}{m_1+m_2}|\mathbf{u}_1 - \mathbf{u}_2|^2 \tag{17}$$

$$= \frac{\frac{1}{2}m_1|\mathbf{u}_1 - \mathbf{u}_2|^2}{1 + \frac{m_1}{m_2}} \tag{18}$$

This makes sense in the limit $m_2 \gg m_1$. Imagine throwing a 100g ball of plasticine on the ground. $m_2$ is the Earth at $5.97\times10^{24}$ kg. If we choose a reference frame fixed to the ground $\mathbf{u}_2 = \mathbf{0}$. The loss in kinetic energy is therefore $\Delta E = \frac{1}{2}m_1|\mathbf{u}_1|^2$. i.e. the entire pre-collision amount associated with the plasticine.

### 6.2 Special case #2: Elastic collision. i.e. $C = 1$

Setting $C = 1$ in Equation 8:

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{u}_1 \left\{ \frac{2m_1}{m_1+m_2} - 1 \right\} + \mathbf{u}_2 \frac{2m_2}{m_1+m_2} \\ \mathbf{v}_2 &= \mathbf{u}_1 \frac{2m_1}{m_1+m_2} + \mathbf{u}_2 \left\{ \frac{2m_2}{m_1+m_2} - 1 \right\} \end{aligned} \tag{19}$$

The total kinetic energy post-impact is:

$$\begin{aligned} T_{post} &= \frac{1}{2} m_1 |\mathbf{v}_1|^2 + \frac{1}{2} m_2 |\mathbf{v}_2|^2 \\ &= \frac{1}{2} m_1 \left| \mathbf{u}_1 \left\{ \frac{2m_1}{m_1+m_2} - 1 \right\} + \mathbf{u}_2 \frac{2m_2}{m_1+m_2} \right|^2 \\ &\quad + \frac{1}{2} m_2 \left| \mathbf{u}_1 \frac{2m_1}{m_1+m_2} + \mathbf{u}_2 \left\{ \frac{2m_2}{m_1+m_2} - 1 \right\} \right|^2 \\ &= \frac{1}{2} m_1 \left| \mathbf{u}_1 \left\{ \frac{m_1-m_2}{m_1+m_2} \right\} + \mathbf{u}_2 \frac{2m_2}{m_1+m_2} \right|^2 \\ &\quad + \frac{1}{2} m_2 \left| \mathbf{u}_1 \frac{2m_1}{m_1+m_2} + \mathbf{u}_2 \left\{ \frac{m_2-m_1}{m_1+m_2} \right\} \right|^2 \\ &= \frac{1}{2} m_1 \left( |\mathbf{u}_1|^2 \left\{ \frac{m_1-m_2}{m_1+m_2} \right\}^2 + |\mathbf{u}_2|^2 \frac{4m_2^2}{(m_1+m_2)^2} + 4\mathbf{u}_1 \cdot \mathbf{u}_2 \left\{ \frac{m_1-m_2}{m_1+m_2} \right\} \frac{m_2}{m_1+m_2} \right) \\ &\quad + \frac{1}{2} m_2 \left( |\mathbf{u}_2|^2 \left\{ \frac{m_1-m_2}{m_1+m_2} \right\}^2 + |\mathbf{u}_1|^2 \frac{4m_1^2}{(m_1+m_2)^2} - 4\mathbf{u}_1 \cdot \mathbf{u}_2 \left\{ \frac{m_1-m_2}{m_1+m_2} \right\} \frac{m_1}{m_1+m_2} \right) \\ &= \frac{1}{2} m_1 |\mathbf{u}_1|^2 \left( \left\{ \frac{m_1-m_2}{m_1+m_2} \right\}^2 + \frac{4m_1 m_2}{(m_1+m_2)^2} \right) + \frac{1}{2} m_2 |\mathbf{u}_2|^2 \left( \frac{4m_1 m_2}{(m_1+m_2)^2} + \left\{ \frac{m_1-m_2}{m_1+m_2} \right\}^2 \right) \\ &= \frac{1}{2} m_1 |\mathbf{u}_1|^2 + \frac{1}{2} m_2 |\mathbf{u}_2|^2 \end{aligned} \tag{20}$$

Since $\left\{ \frac{m_1-m_2}{m_1+m_2} \right\}^2 + \frac{4m_1m_2}{(m_1+m_2)^2} = \frac{m_1^2+m_2^2-2m_1m_2+4m_1m_2}{(m_1+m_2)^2} = \frac{m_1^2+m_2^2+2m_1m_2}{(m_1+m_2)^2} = \frac{(m_1+m_2)^2}{(m_1+m_2)^2} = 1$

Hence in an elastic collision, kinetic energy is conserved. i.e. $\Delta E = T_{pre} - T_{post} = 0$

### 6.3 An interesting result!

Consider an elastic collision where $m_2 \gg m_1$. Using 19:

$$\begin{aligned} \mathbf{v}_1 &= 2\mathbf{u}_2 - \mathbf{u}_1 \\ \mathbf{v}_2 &= \mathbf{u}_2 \end{aligned} \tag{21}$$

If masses are dropped together from height $h$ in a uniform gravitational field (with field strength $g$), and the larger mass strikes the ground elastically before meeting the smaller mass

$$\begin{aligned} \mathbf{u}_1 &= -u\widehat{\mathbf{x}} \\ \mathbf{u}_2 &= u\widehat{\mathbf{x}} \end{aligned} \tag{22}$$

Where $\widehat{\mathbf{x}}$ is a unit vector in the 'up' direction.

Therefore if $\mathbf{v}_1 = v_1\widehat{\mathbf{x}}$ and $\mathbf{v}_2 = v_2\widehat{\mathbf{x}}$

$$\begin{aligned} v_1 &= 3u \\ v_2 &= u \end{aligned} \tag{23}$$

By conservation of energy we can calculate the height risen by both masses following collision. i.e. the height at which the velocity of each mass becomes zero

$$\begin{aligned} m_1 gH &= \frac{1}{2}mv_1^2 \\ m_2 gh &= \frac{1}{2}m_2 u^2 = \frac{1}{2}m_2 v_2^2 \end{aligned} \tag{24}$$

Hence $H = 9\frac{u^2}{2g}$ and $h = \frac{u^2}{2g}$, which yields the fascinating result

$$H = 9h \tag{25}$$

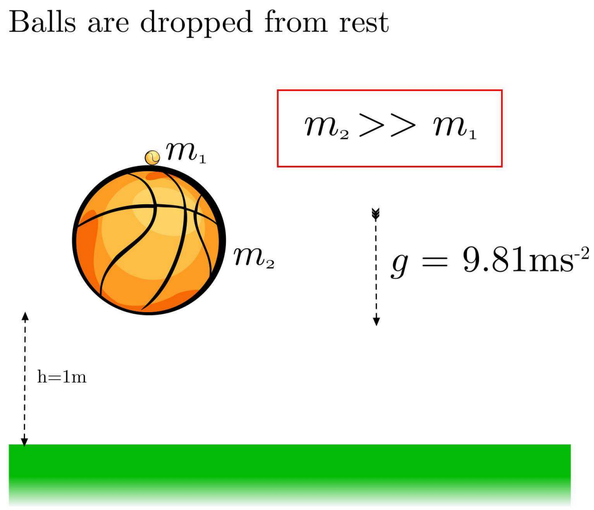


Figure 5: Elastic balls of significantly different mass (e.g. a basket ball and a ping pong ball) are dropped together from height $h$.

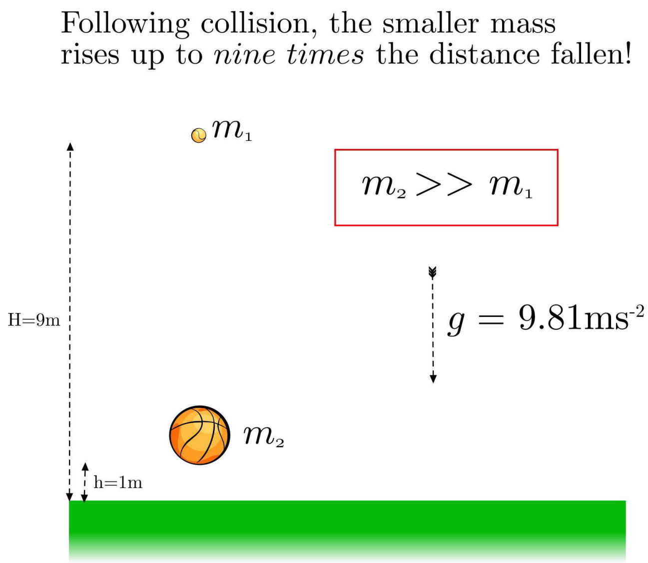


Figure 6: An interesting result! In the limit $m_2 \gg m_1$, two balls dropped together (with the least massive uppermost) will cause the lighter mass to rise up to nine times the original height, if all collisions are perfectly elastic.

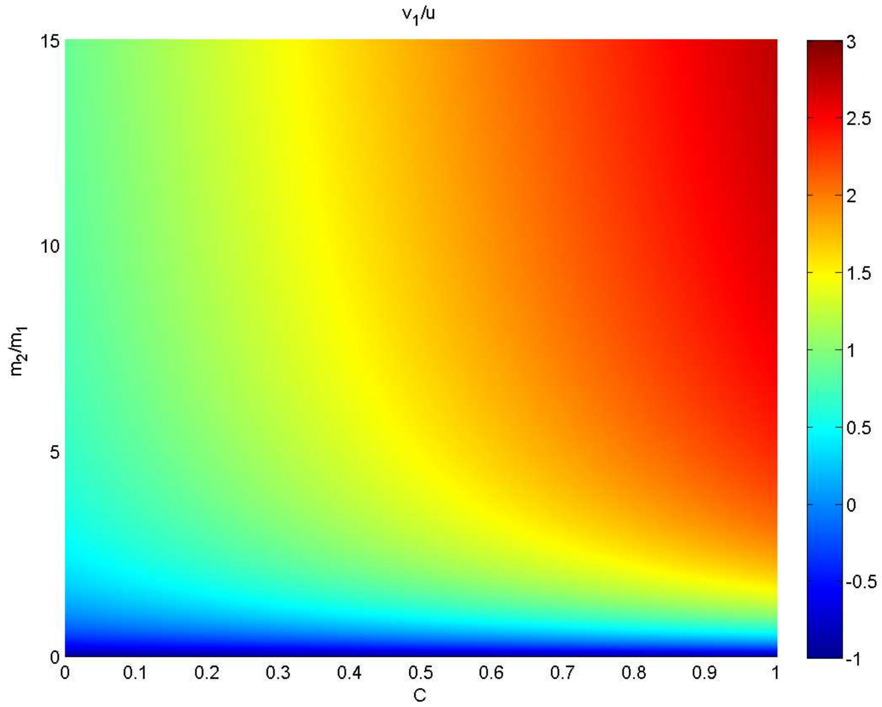


Figure 7: Recoil velocity of mass #1 plotted against coefficient of restitution $C$ and mass ratio $\frac{m_2}{m_1}$. Approach speeds of both masses are $u$. Note for elastic collisions in the limit $m_2 \gg m_1$ the recoil velocity tends to $3u$.

## 7 Graphical exploration

The variation of $\mathbf{v}_1, \mathbf{v}_2, \frac{T_{pre}-T_{post}}{T_{pre}}$ with $C$ and $\frac{m_2}{m_1}$ can be explored graphically using plotting software such as MATLAB and the following simplifications:

$$\mathbf{u}_1 = -\alpha u \widehat{\mathbf{x}} \tag{26}$$
$$\mathbf{u}_2 = u \widehat{\mathbf{x}}$$
$$\mathbf{v}_1 = v_1 \widehat{\mathbf{x}} \tag{27}$$
$$\mathbf{v}_2 = v_2 \widehat{\mathbf{x}} \tag{28}$$

where $\widehat{\mathbf{x}}$ is a unit vector and $\alpha$ is a scalar parameter. For the plots in Figures 7 to 9, $\alpha = 1$. The general results in equations 8, 10 and 9 simplify to

$$v_1 = \frac{u}{m_1 + m_2} \left( -\alpha \left( m_1 - C m_2 \right) + m_2 (1 + C) \right) \tag{29}$$
$$v_2 = \frac{u}{m_1 + m_2} \left( m_2 - C m_1 - \alpha m_1 (1 + C) \right)$$
$$T_{pre} = \frac{1}{2} u^2 \left( m_1 \alpha^2 + m_2 \right) \tag{30}$$
$$T_{post} = \frac{m_1 m_2 u^2}{2 \left( m_1 + m_2 \right)} \left\{ A\alpha^2 + B - 2\alpha (1 - C^2) \right\} \tag{31}$$

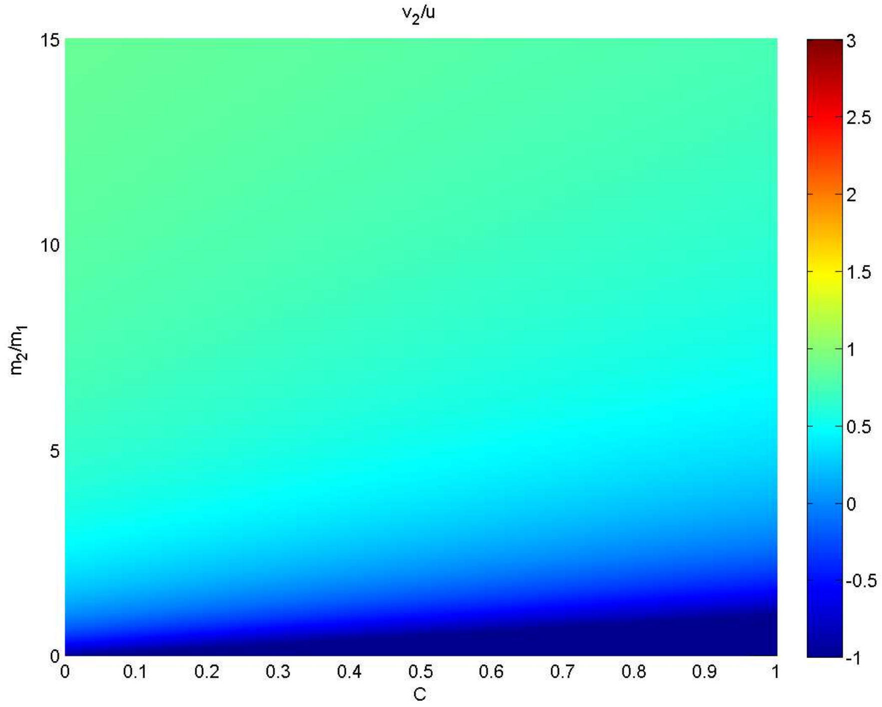


Figure 8: Recoil velocity of mass #2 plotted against coefficient of restitution $C$ and mass ratio $\frac{m_2}{m_1}$. Approach speeds of both masses are $u$.

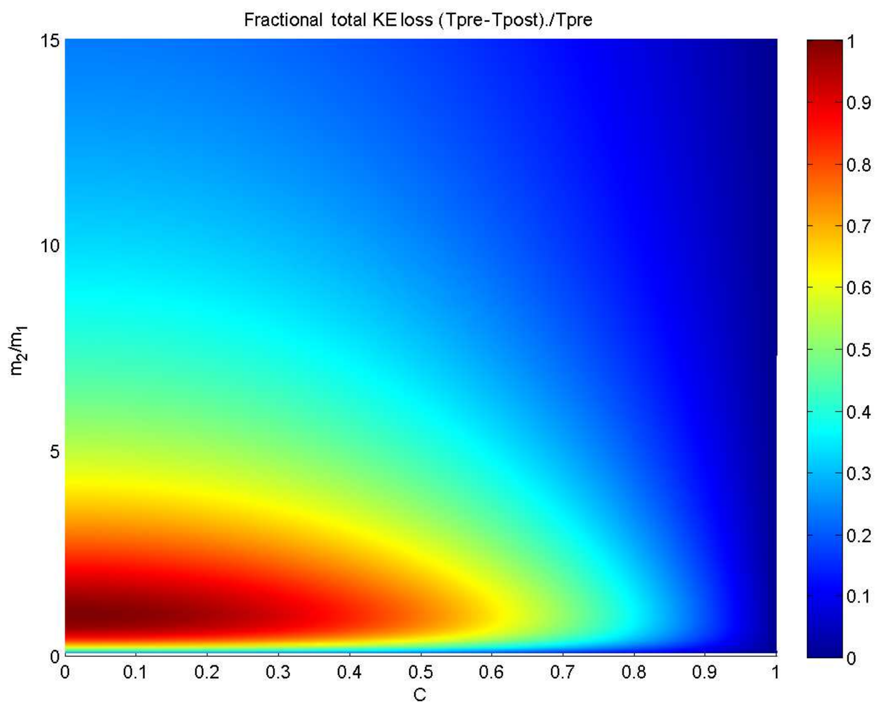


Figure 9: Fractional loss of total kinetic energy (KE) of two masses following collision. Both masses approach each other with speed $u$. For an inelastic collision ($C = 0$) the maximum loss (100%) of KE is for $m_1 = m_2$. For elastic ($C = 1$) collisions there is no loss of KE.

# 8 Extension: 'The Irish Moon Shot'

The surprising result that the elastic collision of two spheres results in a speed multiplication (of the least massive) by up to three times (in the limit that the heavier sphere is infinitely heavier) can be used to explore the following question:

"$N$ elastic spheres are stacked and dropped as an ensemble from a height $h$ of one metre, reaching a velocity $u = \sqrt{2gh}$. The uppermost sphere has mass $m$ and subsequent spheres increase in mass by a constant factor $k$. If all spheres collide with coefficient of restitution $C$ and the inter-sphere collisions can be modelled seperately[2], how many spheres are needed to cause the uppermost mass to escape the Earth's gravitational field?"

*Solution*

Consider the collision between mass $m_n$ and $m_{n+1}$. If $\widehat{\mathbf{x}}$ is a unit vector pointing up (i.e. in the opposite direction to the local gravitational field $\mathbf{g}$) the pre-collision velocities are:

$$\begin{aligned} \mathbf{u}_n &= -u\widehat{\mathbf{x}} \\ \mathbf{u}_{n+1} &= v_{n+1}\widehat{\mathbf{x}} \end{aligned} \tag{32}$$

The post-collision velocity of mass $m_n$ is $\mathbf{v}_n = v_n\widehat{\mathbf{x}}$. The general result in equation 8 states

$$\mathbf{v}_n = \mathbf{u}_n \left\{ \frac{m_n - Cm_{n+1}}{m_n + m_{n+1}} \right\} + \mathbf{u}_{n+1} \frac{m_{n+1}(1+C)}{m_n + m_{n+1}} \tag{33}$$

Hence:

$$v_n = \frac{-m_n + Cm_{n+1}}{m_n + m_{n+1}} u + v_{n+1} \frac{m_{n+1}(1+C)}{m_n + m_{n+1}} \tag{34}$$

Now $\frac{m_{n+1}}{m_n} = k$ , therefore

$$v_n = \frac{kC-1}{1+k} u + v_{n+1} \frac{k(1+C)}{1+k} \tag{35}$$

Which simplifies to

$$\begin{aligned} v_{n+1} &= \frac{1+k}{k(1+C)} v_n - \frac{kC-1}{k(1+C)} u \\ &= av_n - b \end{aligned} \tag{36}$$

Let us consider the first few terms

$$\begin{aligned} v_2 &= av_1 - b \\ v_3 &= av_2 - b = a\left(av_1 - b\right) - b = a^2 v_1 - ab - b \\ v_4 &= av_3 - b = a\left(a^2 v_1 - ab - b\right) - b = a^3 v_1 - a^2 b - ab - b \\ v_5 &= av_4 - b = a^4 v_1 - a^3 b - a^2 b - ab - b \end{aligned} \tag{37}$$

By spotting the *geometric* pattern we can write (for $n > 1$)

$$v_n = a^{n-1} v_1 - b \sum_{y=0}^{n-2} a^y \tag{38}$$

This summation can be evaluated using the standard result

$$S_n = a \sum_{i=0}^{n-1} r^i = a \frac{1 - r^n}{1 - r} \tag{39}$$

[2]Rather than a multi body collision, which may not have an analytic solution.

Hence

$$\sum_{y=0}^{n-2} a^y = \frac{1-a^{n-1}}{1-a} \tag{40}$$

which gives

$$v_n = a^{n-1}v_1 - b\frac{1-a^{n-1}}{1-a} \tag{41}$$

Now mass $v_N$ collides with the Earth with pre-collision velocity $\mathbf{u}_N = -u\widehat{\mathbf{x}}$ and therefore rebounds with velocity $\mathbf{v}_N = uC\widehat{\mathbf{x}}$. Therefore

$$uC = a^{N-1}v_1 - b\frac{1-a^{N-1}}{1-a} \tag{42}$$

which can be rearranged to yield an expression for the recoil velocity $v_1$ of the uppermost mass.

$$v_1 = \frac{uC}{a^{N-1}} + b\frac{1-a^{N-1}}{(1-a)\,a^{N-1}} \tag{43}$$

$$= uCa^{1-N} + b\frac{\left(a^{1-N}-1\right)}{1-a} \tag{44}$$

Since $a = \frac{1+k}{k(1+C)}$ and $b = \frac{kC-1}{k(1+C)}u$ this gives

$$\begin{aligned}
\frac{v_1}{u} &= C\left(\frac{1+k}{k(1+C)}\right)^{1-N} + \frac{kC-1}{k(1+C)}\frac{\left(\frac{1+k}{k(1+C)}\right)^{1-N}-1}{\left(1-\frac{1+k}{k(1+C)}\right)} \\
&= C\left(\frac{1+k}{k(1+C)}\right)^{1-N} + \frac{kC-1}{k(1+C)}\frac{\left(\frac{1+k}{k(1+C)}\right)^{1-N}-1}{\frac{k(1+C)-1-k}{k(1+C)}} \\
&= C\left(\frac{1+k}{k(1+C)}\right)^{1-N} + \left(\frac{1+k}{k(1+C)}\right)^{1-N} - 1
\end{aligned}$$

Hence

$$\frac{v_1}{u} = \left(\frac{k(1+C)}{1+k}\right)^{N-1}(C+1) - 1$$

Now consider the special case when $k = 2$ and $C = 1$

$$\frac{v_1}{u} = 2\left(\frac{4}{3}\right)^{N-1} - 1$$

Now for mass $m_1$ to escape the Earth's gravitational field

$$v_1 = \sqrt{\frac{2GM}{R_E}}$$

where $G = 6.67\times10^{-11}\mathrm{m^3kg^{-1}s^{-2}}$, $M = 5.97\times10^{24}$kg and $R_E = 6.38\times10^6$m. Therefore if $u = \sqrt{2gh}$ where $h$ is the initial height dropped ($h =$1m in our example)

$$\sqrt{\frac{GM}{R_E gh}} = 2\left(\frac{4}{3}\right)^{N-1} - 1$$

Which implies

$$N = \text{ceil}\left\{1 + \frac{\log\left(\sqrt{\frac{GM}{R_E gh}} + 1\right) - \log 2}{\log \frac{4}{3}}\right\} = 26$$

where 'ceil' rounds up (25.82) to the nearest integer.

So is this a practical moonshot? If the uppermost mass is 1kg this means the $26^{\text{th}}$ mass is $2^{25}$kg $\approx 33,554$ metric tonnes. This implies some very large elastic balls!

Perhaps  more practical example is to consider $k = 2$, $N = 4$ and $C = 1$. This is possibly characteristic of the Irish Moonshot toy avaliable from good science retailers. In this case $\frac{v_1}{u} = 2\left(\frac{4}{3}\right)^3 - 1 = \frac{128}{27} - 1 \approx 3.7$.