1 Sketching $y = x^x$

In order to sketch the function $y = x^x$ let us first re-write it using the idenity $x \equiv e^{\ln x}$

$$x^x = \left(e^{\ln x}\right)^x = e^{x \ln x} \tag{1}$$

A series expansion of x^x is therefore the expansion of e^x but with $x \ln x$ substituted for x. The Maclaurin Expansion for $f(x)$ is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + ... = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \tag{2}$$

If $f(x) = e^x$

$$f^{(n)}(x) = e^x \tag{3}$$

Hence

$$f^{(n)}(0) = 1 \tag{4}$$

Therefore

$$e^x = 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + ... \tag{5}$$

$$e^x = \sum_{n=0}^{\infty} \tfrac{1}{n!}x^n \tag{6}$$

Hence we can write x^x in terms of an infinite series

$$x^x = e^{x \ln x} = 1 + x \ln x + \tfrac{1}{2!}(x \ln x)^2 + \tfrac{1}{3!}(x \ln x)^3 + ... \tag{7}$$

$$x^x = \sum_{n=0}^{\infty} \tfrac{1}{n!}(x \ln x)^n \tag{8}$$

This series expansion enables us to clearly show that if $g(x) = x^x$

$$g(0) = 1 \tag{9}$$

$$g(1) = 1 \tag{10}$$

since $\ln(1) = 0$.

The first derivative of $y = x^x$ can proceed directly from $x^x = e^{x \ln x}$ using the chain and product rules:

$$\frac{dy}{dx} = e^{x \ln x}\left(x\frac{d}{dx}\ln x + \ln x\right) \tag{11}$$

$$\frac{dy}{dx} = x^x(1 + \ln x) \tag{12}$$

An alternative is to firstly take natural logarithms of $y = x^x$ and differentiate implicitly:

$$y = x^x \tag{13}$$

$$\ln y = x \ln x \tag{14}$$

$$\frac{1}{y}\frac{dy}{dx} = x\frac{d}{dx}\ln x + \ln x \tag{15}$$

$$\frac{dy}{dx} = x^x(1 + \ln x) \tag{16}$$

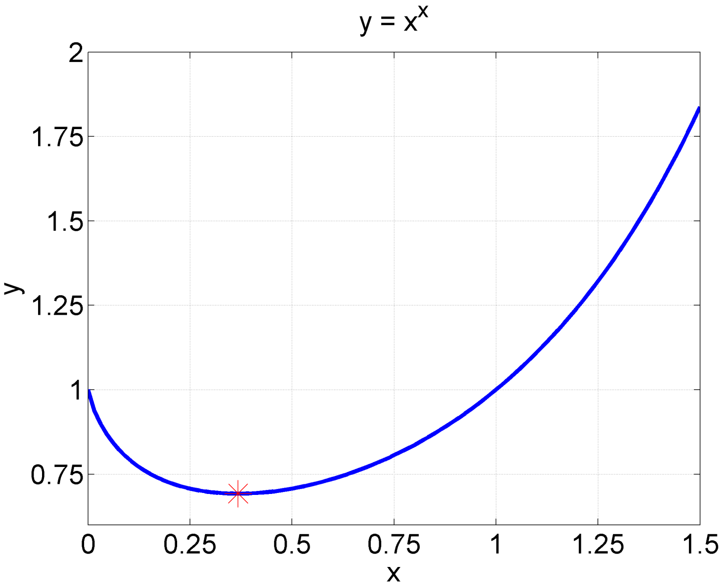

Figure 1: Plot of the function $y = x^x$. This function has y values of unity when $x = 0$ and $x = 1$. The minimum is at stationary point $\left(\frac{1}{e}, \frac{1}{e^{\frac{1}{e}}}\right)$.

Since $\ln x$ is negative in the range $0 < x < 1$, we expect a minimum of $y = x^x$ within $0 < x < 1$. A rapid increase is anticipated for $x > 1$ since both x^x and $\ln x$ are positive and increasing. Since $x^x > 0$ in the range $0 < x < 1$, the stationary point, when $\frac{dy}{dx} = 0$, is for x given by

$$1 + \ln x = 0 \tag{17}$$
$$\ln x = -1 \tag{18}$$
$$x = \frac{1}{e} \tag{19}$$

Hence the stationary point of $y = x^x$ is $\left(\frac{1}{e}, \frac{1}{e^{\frac{1}{e}}}\right)$

Note since $y = x^x$ is an infinite series involving $\ln x$, this means it is undefined for $x < 0$, unless one considers an extension of the meaning of $\ln x$ into the complex domain.

A plot of $y = x^x$ which illustrates all of these features is provided in Figure 1.

2 Integrating $y = x^x$

The series expansion of $y = x^x$ is useful to enable the evaluation of its anti-differential, and hence definite integrals of the curve $y = x^x$.

$$\int_a^b x^x dx = \int_a^b e^{x \ln x} dx = \int_a^b \sum_{n=0}^{\infty} \tfrac{1}{n!} (x \ln x)^n dx \tag{20}$$

$$\int_a^b x^x dx = \sum_{n=0}^{\infty} \tfrac{1}{n!} \int_a^b (x \ln x)^n dx \tag{21}$$

where limits $a, b \geq 0$

To evaluate the integral above, let us firstly define a more general definite integral:

$$I_{m,n} = \int_a^b x^m (\ln x)^n dx \tag{22}$$

where $m, n \geq 0$ and are integers. This integral can be evaluated by-parts:

$$I_{m,n} = \left[\frac{x^{m+1}}{m+1} (\ln x)^n \right] - \int_a^b \frac{x^{m+1}}{m+1} \frac{n (\ln x)^{n-1}}{x} dx \tag{23}$$

using the results

$$\int_a^b uv dx = \left[\left(\int u dx \right) v \right]_a^b - \int_a^b \left(\int u dx \right) \frac{dv}{dx} dx \tag{24}$$

and

$$\frac{d}{dx} (\ln x)^n = \frac{n (\ln x)^{n-1}}{x} \tag{25}$$

Hence

$$I_{m,n} = \left[\frac{x^{m+1}}{m+1} (\ln x)^n \right]_a^b - \frac{n}{m+1} \int_a^b x^m (\ln x)^{n-1} dx \tag{26}$$

$$I_{m,n} = \left[\frac{x^{m+1}}{m+1} (\ln x)^n \right]_a^b - \frac{n}{m+1} I_{m,n-1} \tag{27}$$

If we take the limits $a = 0$ and $b = 1$ (the integral $I_{m,n}$ will assume this from now on)

$$\left[\frac{x^{m+1}}{m+1} (\ln x)^n \right]_0^1 = 0 \tag{28}$$

Hence we can express $I_{m,n}$ via Reduction Formulae in terms of $I_{m,n-1}$ and ultimately (via repeated iterative substitution), $I_{m,0}$.

$$I_{m,n} = -\frac{n}{m+1} I_{m,n-1} \tag{29}$$

$$I_{m,n} = \left(-\frac{n}{m+1} \right) \left(-\frac{n-1}{m+1} \right) \left(-\frac{n-2}{m+1} \right) .... \left(-\frac{1}{m+1} \right) I_{m,0} \tag{30}$$

$$I_{m,n} = \frac{(-1)^n n!}{(m+1)^n} I_{m,0} \tag{31}$$

Now the $n = 0$ term can be evaluated explicitly:

$$I_{m,0} = \int_0^1 x^m (\ln x)^0 dx = \int_0^1 x^m dx = \frac{1}{m+1} \tag{32}$$

Therefore

$$I_{m,n} = \frac{(-1)^n \, n!}{(m+1)^{n+1}} \tag{33}$$

This enables us to write down a closed form expression for the integral of $y = x^x$ over the interval [0,1]

$$\begin{aligned}
\int_0^1 x^x dx &= \sum_{n=0}^{\infty} \tfrac{1}{n!} \int_1^0 (x \ln x)^n \, dx && (34) \\
\int_0^1 x^x dx &= \sum_{n=0}^{\infty} \tfrac{1}{n!} I_{n,n} && (35) \\
\int_0^1 x^x dx &= \sum_{n=0}^{\infty} \tfrac{1}{n!} \frac{(-1)^n \, n!}{(n+1)^{n+1}} && (36) \\
\int_0^1 x^x dx &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^{n+1}} && (37)
\end{aligned}$$

i.e.

$$\int_0^1 x^x dx = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + ... \tag{38}$$

AF (with inspiration from CHJH) 4/6/2014.